Ramsey numbers for line graphs and perfect graphs

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joint work with

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COCOON 2012

Sydney, Australia

August 20–22, 2012

Ramsey numbers

Definition

For any pair of positive integers (i, j), the *Ramsey number* $R(i, j)$ is the smallest integer p such that every graph on p vertices contains a clique of size i or an independent set of size j.

Theorem (Ramsey; 1930)

For every pair of positive integers (i, j), the number $R(i, j)$ exists.

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Observation

$R(i, j) = R(j, i)$ *for every pair of positive integers* (i, j).

Known (bounds on) Ramsey numbers

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$R(1, j) = 1$ *and* $R(2, j) = j$ *for every integer* $j \geq 1$.

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4	9	18	25	35-41	49-61	56-84	73-115	92-149
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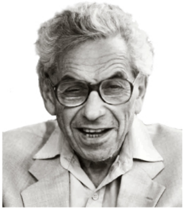

"Imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5,5)$ *or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value."*

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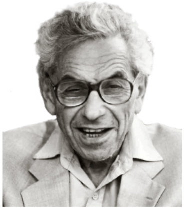

"But suppose, instead, that they ask for $R(6, 6)$. In that case, we should attempt to destroy the aliens."

Ramsey numbers for graph classes

Definition

Let $\mathcal{G}$ be a class of graphs. For any pair of positive integers (i,j), the *Ramsey number* $R_{\mathcal{G}}(i,j)$ is the smallest integer p such that every graph on p vertices that belongs to $\mathcal{G}$ contains a clique of size i or an independent set of size j.

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For any graph class $\mathcal{G}$ *and every pair of positive integers* (i,j), $R_{\mathcal{G}}(i,j) \leq R(i,j)$.

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For any graph class $\mathcal{G}$ and every pair of positive integers (i, j), $R_{\mathcal{G}}(i, j) \leq R(i, j)$.

Goal

Identify graph classes $\mathcal{G}$ for which the value $R_{\mathcal{G}}(i, j)$ can be determined for every pair of positive integers (i, j).

Known results on Ramsey numbers for graph classes

Theorem (Walker; 1969)

Let $\mathcal{P}$ be the class of planar graphs. Then

- $R_{\mathcal{P}}(2, j) = j$ *for* $j \geq 1$ *and* $R_{\mathcal{P}}(i, 2) = i$ *for* $i \leq 5$*;*
- $R_{\mathcal{P}}(3, j) = 3j - 3$ *for* $j \geq 2$*;*
- $4j - 3 \leq R_{\mathcal{P}}(i, j) \leq 5j - 4$ *for* $i \geq 4$ *and* $(i, j) \neq (4, 2)$*.*

Moreover, the truth of the four-color conjecture would imply that

- $R_{\mathcal{P}}(i, j) = 4j - 3$ *for* $i \geq 4$ *and* $(i, j) \neq (4, 2)$*.*

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Theorem (Steinberg & Tovey; JCTB 1993)

The planar Ramsey numbers are:

- $R_{\mathcal{P}}(2, j) = j$ *for* $j \geq 1$ *and* $R_{\mathcal{P}}(i, 2) = i$ *for* $i \leq 5$*.*
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Let $\mathcal{C}$ be the class of claw-free graphs.

Theorem (Matthews; 1985)

$R_{\mathcal{C}}(i,3) = R(i,3)$ *for every positive integer* i.

Proof.

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Proof. $R_{\mathcal{C}}(i, 3) \leq R(i, 3)$ by definition.

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Recall that $R(i,3)$ is unknown for any $i \geq 10$.

Known results

We have a general formula for $R_{\mathcal{G}}(i,j)$ if $\mathcal{G}$ is the class of:

- planar graphs

There is no hope in finding such a formula if $\mathcal{G}$ is the class of:

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Known results and our results

We have a general formula for $R_{\mathcal{G}}(i, j)$ if $\mathcal{G}$ is the class of:

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- perfect graphs[1]

[1] as well as any subclass of perfect graphs that contains disjoint unions of complete graphs, such as chordal graphs, interval graphs, proper interval graphs, permutation graphs, cocomparability graphs, and cographs.

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- split graphs
- threshold graphs
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Theorem

$R_{\mathcal{P}}(i,j) = (i-1)(j-1)+1$ *for each pair of positive integers* (i,j).

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Proof. It is easy to see that $R_{\mathcal{P}}(i,j) \geq (i-1)(j-1)+1$.
(Consider the graph consisting of $j-1$ disjoint copies of K_{i-1}.)

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Proof. Let G be a perfect graph on $(i-1)(j-1)+1$ vertices. Suppose G has no K_i.

- $\omega(G) \leq i-1$
- $\chi(G) \leq i-1$

Consider an optimal coloring φ of G.

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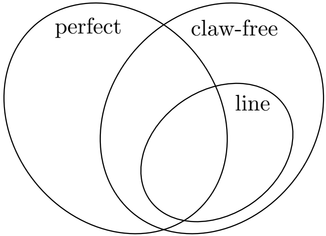
Ramsey numbers for line graphs

Definition

Let H be a graph. The *line graph* of H, denoted $L(H)$, is the graph with vertex set $E(H)$, such that there is an edge between two vertices e, e' if and only if the edges e and e' are incident in H.

If $G = L(H)$ is a line graph, then H is the *preimage* graph of G.



Ramsey numbers for line graphs

Let $\mathcal{L}$ be the class of line graphs.

Observation

For every integer $j \geq 1$, $R_{\mathcal{L}}(1,j) = 1$ *and* $R_{\mathcal{L}}(2,j) = j$.

Theorem

For every integer $j \geq 1$,

$$R_{\mathcal{L}}(3,j) = \begin{cases} \frac{5(j-1)-1}{2} + 1 & \text{if } j \text{ is even,} \\ \frac{5(j-1)}{2} + 1 & \text{if } j \text{ is odd.} \end{cases}$$

Ramsey numbers for line graphs

Main Theorem

For every pair of integers $i \geq 4$ *and* $j \geq 1$,

$$R_{\mathcal{L}}(i,j) = \begin{cases} i(j-1) - (t+r) + 2 & \text{if } i = 2k, \\ i(j-1) - r + 2 & \text{if } i = 2k+1, \end{cases}$$

where $j = tk + r$, $t \geq 0$ *and* $1 \leq r \leq k$.

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where $j = tk + r$, $t \geq 0$ *and* $1 \leq r \leq k$.

For convenience, we define the following function β:

$$\beta(i,j) = \begin{cases} i(j-1) - (t+r) + 1 & \text{if } i = 2k, \\ i(j-1) - r + 1 & \text{if } i = 2k+1, \end{cases}$$

where $j = tk + r$, $t \geq 0$ and $1 \leq r \leq k$.

Main Theorem (alternative formulation)

For every pair of integers $i \geq 4$ *and* $j \geq 1$, $R_{\mathcal{L}}(i,j) = \beta(i,j) + 1$.

Ramsey numbers for line graphs

Well-known fact

Every line graph other than K_3 has a unique preimage graph.

Observation

Let G be a line graph, and let H be its preimage graph, i.e., $G = L(H)$. For $i \geq 4$ and $j \geq 1$, the following holds:

- *G contains a clique of size i if and only if H contains a vertex of degree i;*
- *G contains an independent set of size j if and only if H contains a matching of size j.*

Ramsey numbers for line graphs: upper bound

Theorem

Let $i \geq 4$ and $j \geq 1$ be two integers, and let H be a graph that has no vertex of degree $\geq i$ and no matching of size $\geq j$. Then H has at most $\beta(i, j)$ edges.

Proof. We use induction on j.

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Corollary

Let $i \geq 4$ and $j \geq 1$ be two integers, and let G be a line graph that has no clique of size $\geq i$ and no independent set of size $\geq j$. Then G has at most $\beta(i, j)$ vertices.

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Corollary (upper bound)

For every pair of integers $i \geq 4$ and $j \geq 1$, $R_{\mathcal{L}}(i,j) \leq \beta(i,j) + 1$.

Ramsey numbers for line graphs: lower bound

Theorem

For every pair of integers $i \geq 4$ and $j \geq 1$, there exists a graph H with $\beta(i, j)$ edges that has no vertex of degree $\geq i$ and no matching of size $\geq j$.

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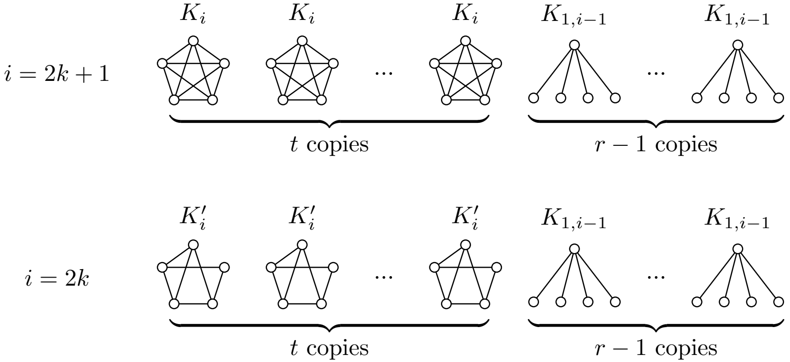

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Corollary

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Ramsey numbers for line graphs: lower bound

Theorem

For every pair of integers $i \geq 4$ *and* $j \geq 1$, *there exists a graph* H *with* $\beta(i,j)$ *edges that has no vertex of degree* $\geq i$ *and no matching of size* $\geq j$.

Corollary

For every pair of integers $i \geq 4$ *and* $j \geq 1$, *there exists a line graph* G *on* $\beta(i,j)$ *vertices that has no clique of size* $\geq i$ *and no independent set of size* $\geq j$.

Corollary (lower bound)

For every pair of integers $i \geq 4$ *and* $j \geq 1$, $R_{\mathcal{L}}(i,j) \geq \beta(i,j) + 1$.

Ramsey numbers for line graphs: tight bound

Theorem

For every pair of integers $i \geq 4$ and $j \geq 1$, there exists a graph H with $\beta(i,j)$ edges that has no vertex of degree $\geq i$ and no matching of size $\geq j$.

Corollary

For every pair of integers $i \geq 4$ and $j \geq 1$, there exists a line graph G on $\beta(i,j)$ vertices that has no clique of size $\geq i$ and no independent set of size $\geq j$.

Theorem

For every pair of integers $i \geq 4$ and $j \geq 1$, $R_{\mathcal{L}}(i,j) = \beta(i,j) + 1$.

Concluding remarks

Goal

Identify graph classes $\mathcal{G}$ for which the value $R_{\mathcal{G}}(i, j)$ can be determined for every pair of positive integers (i, j).

Possible candidates:

- quasi-line graphs
- unit disk graphs
- circle graphs

Concluding remarks

Much, much more difficult tasks:

- Determine the values of $R(5, 5)$ and $R(6, 6)$.

Concluding remarks

Much, much more difficult tasks:

- Determine the values of $R(5,5)$ and $R(6,6)$.
- Alternatively, design a method that can be used to destroy (or seriously confuse) any alien that might ask for these values.

 Dank u wel!

 Takk!

 Cheers!