Edge contractions in subclasses of chordal graphs

Pim van 't Hof
University of Bergen, Norway

joint work with Rémy Belmonte and Pinar Heggernes

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University of Warwick, UK, April 2011

Contractibility

Input: Graphs G and H.

Question: Can G be contracted to H?

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Induced Subgraph Isomorphism

Input: Graphs G and H.
Question: Does G contain H as an induced subgraph?

H-CONTRACTIBILITY

Input: Graph G.
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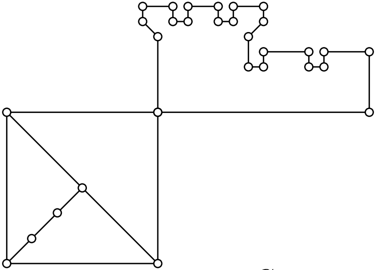
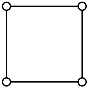
H-INDUCED SUBGRAPH ISOMORPHISM

Input: Graph G.
Question: Does G contain H as an induced subgraph?

Does G contain H as an induced subgraph?



G



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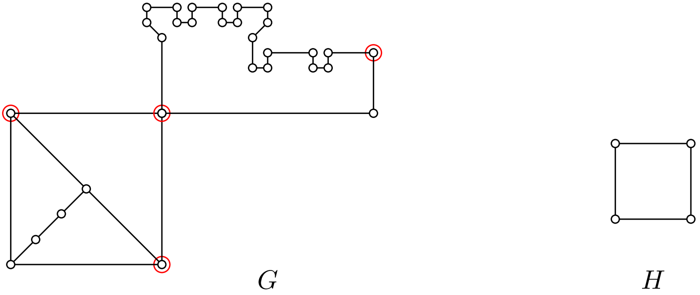

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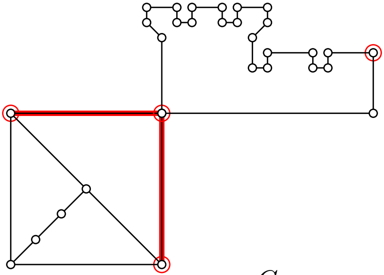

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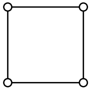

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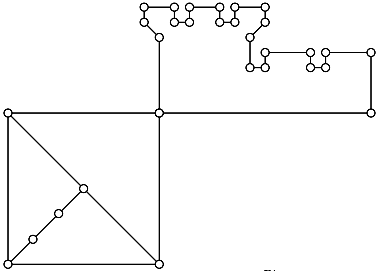

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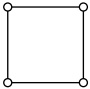

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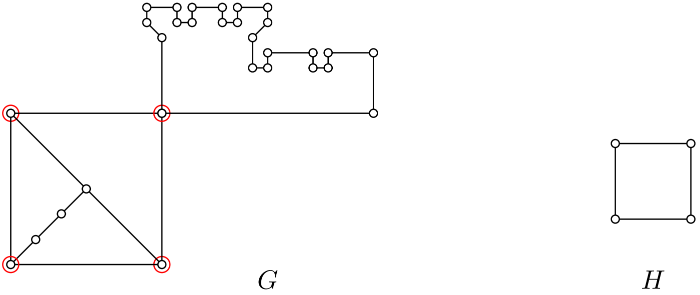

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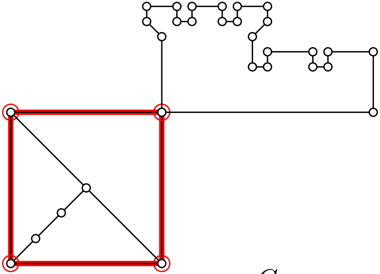

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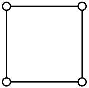

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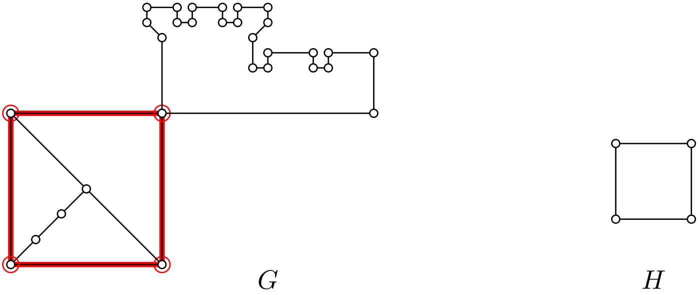
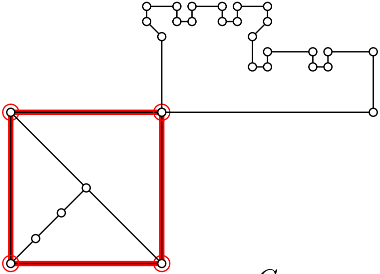


Observation

Induced Subgraph Isomorphism *can be solved in* $f(|V(H)|) \cdot |V(G)|^{|V(H)|}$ *time.*

Does G contain H as an induced subgraph?



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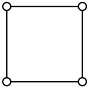

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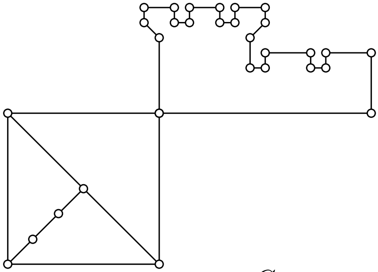
Observation

H-Induced Subgraph Isomorphism *can be solved in polynomial time for every graph* H.

Can G be contracted to H?



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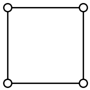

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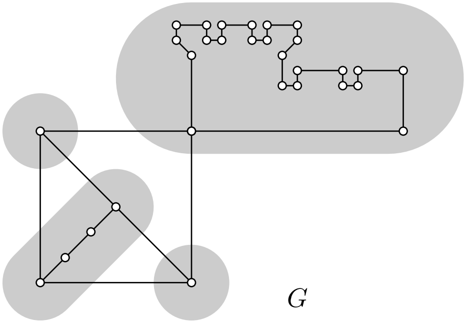

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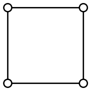

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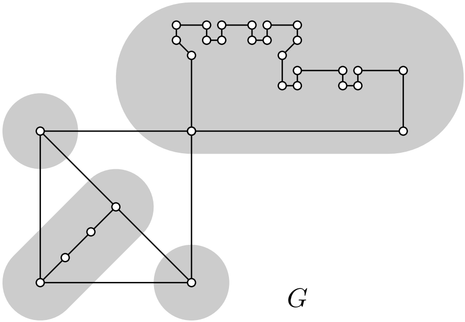

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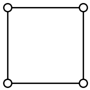

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Note that there are $|V(H)|^{|V(G)|}$ partitions.

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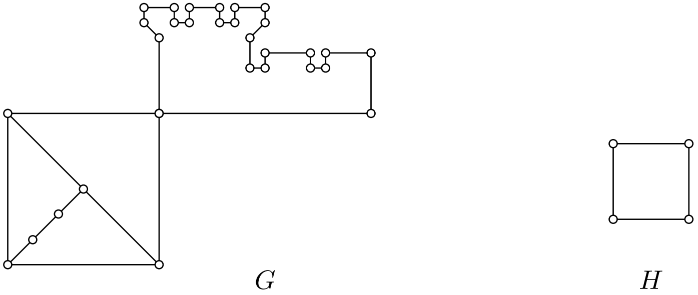

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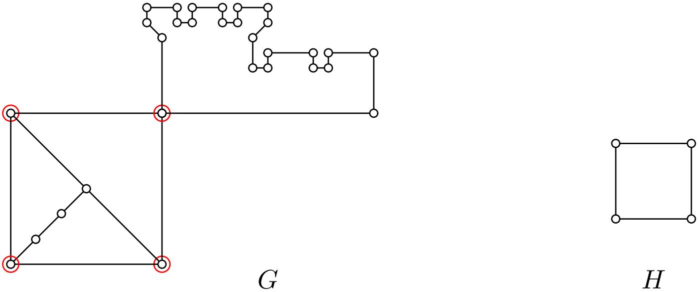

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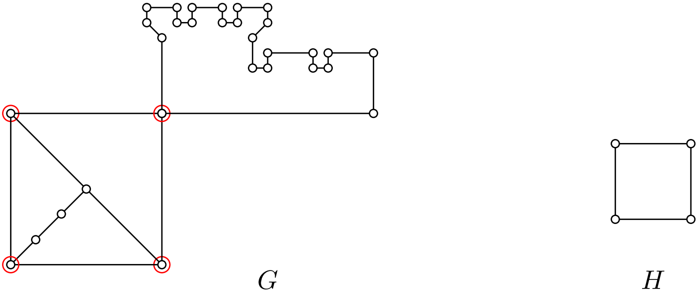

Theorem (Brouwer & Veldman, 1987)

C_4-CONTRACTIBILITY *is NP-complete.*

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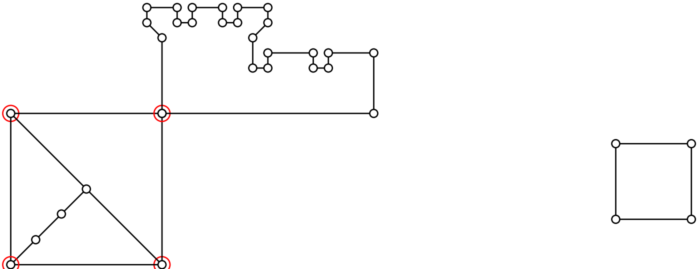

G H

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Can G be contracted to H?

What if both graphs are given as input?

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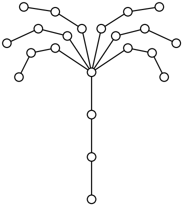

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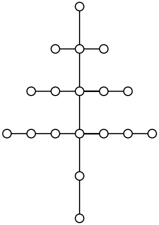

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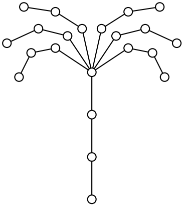

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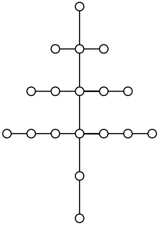

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Theorem (Matoušek & Thomas, 1992)

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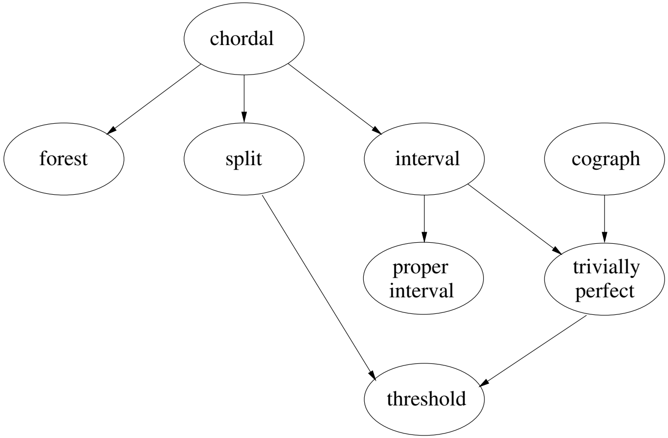
CONTRACTIBILITY *is NP-complete on connected trivially perfect graphs.*


chordal
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Theorem (Wolk, 1962 & 1965)

A graph G is trivially perfect if and only if each connected induced subgraph of G has a universal vertex.

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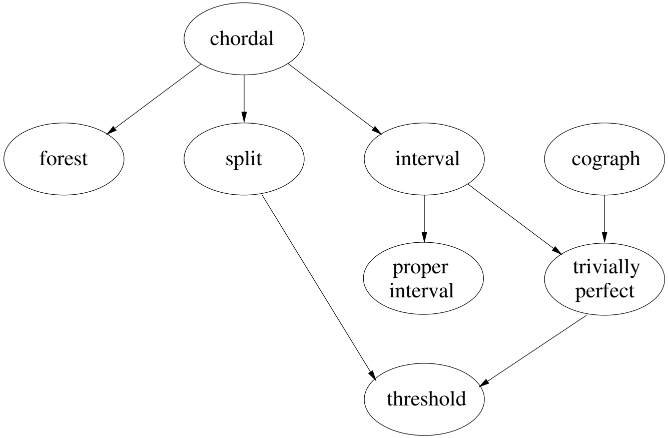
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CONTRACTIBILITY *can be solved in polynomial time if G is trivially perfect and H is threshold.*

Theorem (Damaschke, 1991; Garey & Johnson, 1979)

Induced Subgraph Isomorphism *is NP-complete on disjoint unions of paths.*

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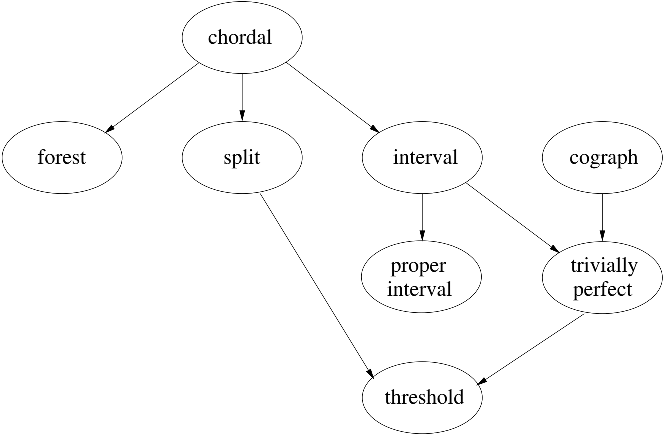
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Observation

For every edge uv of a trivially perfect graph, we have $N[u] \subseteq N[v]$ or $N[v] \subseteq N[u]$.

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Let uv be an edge of a trivially perfect graph G such that $N[u] \subseteq N[v]$. Then G/uv is isomorphic to $G - u$.

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Corollary

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Corollary

Induced Subgraph Isomorphism *can be solved in polynomial time if G is trivially perfect and H is threshold.*

Corollary

Induced Subgraph Isomorphism *can be solved in linear time if G is threshold and H is arbitrary.*

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Corollary

H-Contractibility *can be solved in polynomial time on trivially perfect graphs, for every graph H.*

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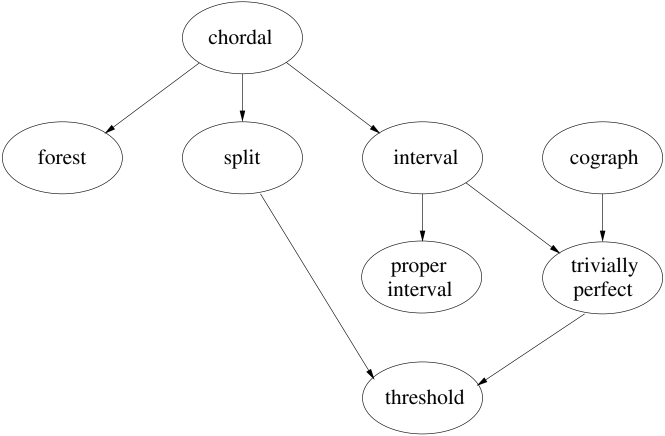
CONTRACTIBILITY *is NP-complete on input pairs* (G, H) *where* G *is a connected split graph and* H *is a connected threshold graph.*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Theorem

H-CONTRACTIBILITY *can be solved in polynomial time on split graphs, for every graph* H.

Lemma

*Let G and H be two split graphs. Then G can be contracted to H if and only if G contains an H-*compatible *set.*

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Given a set $S \subseteq I_G$ of size $|I_H|$, we can test in $f(|V(H)|) \cdot |V(G)|^3$ time if S is H-compatible.

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Theorem

H-CONTRACTIBILITY *can be solved in $f(|V(H)|) \cdot |V(G)|^{3+|V(H)|}$ time on split graphs, for every graph H.*

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Question

Can H-CONTRACTIBILITY be solved in $f(|V(H)|) \cdot |V(G)|^{O(1)}$ time on split graphs?

Theorem

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Question

Is CONTRACTIBILITY *fixed parameter tractable* on split graphs when parameterized by $|V(H)|$?

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Theorem (Golovach, Kamiński, Paulusma & Thilikos, submitted)

CONTRACTIBILITY *is W[1]-hard on split graphs when parameterized by* $|V(H)|$.

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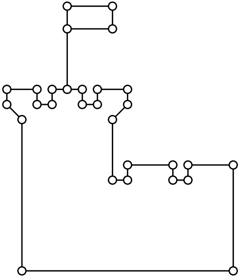
 Dank u wel!

 Thank you!

 Takk!



Pim van 't Hof
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