Contracting graphs to paths and trees

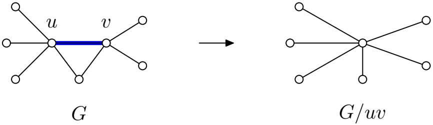
Pim van 't Hof
University of Bergen (Norway)

joint work with

Pinar Heggernes	*University of Bergen (Norway)*
Benjamin Lévêque	*LIRMM, Montpellier (France)*
Daniel Lokshtanov	*U. California, San Diego (USA)*
Christophe Paul	*LIRMM, Montpellier (France)*

IPEC 2011
Saarbrücken, Germany, September 7–9, 2011

The stars of this show



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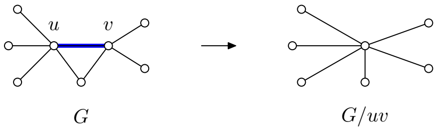

Path Contraction

Input: Graph G, integer k.
Parameter: k.
Question: Is G k-contractible to a path?

Tree Contraction

Input: Graph G, integer k.
Parameter: k.
Question: Is G k-contractible to a tree?

Graph modification problems

problem	vd	ed	ea	ec	target
Vertex Cover	✓				edgeless
Feedback Vertex Set	✓				acyclic
Odd Cycle Transversal	✓				bipartite
Chordal Deletion	✓				chordal
Interval Completion			✓		interval
Minimum Fill-In			✓		chordal
Cluster Editing		✓	✓		P_3-free
Longest Induced Path	✓				path

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Theorem (Asano & Hirata, 1983)

Π-CONTRACTION *is NP-complete for the following classes* Π*:*

- *planar*
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- *without cycles of length at least* ℓ*, for any fixed* $\ell \geq 3$

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Corollary

TREE CONTRACTION *is NP-complete.*

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Theorem (Brouwer & Veldman, 1987)

P_4-CONTRACTIBILITY is NP-complete.

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A graph G is k-contractible to a path if and only if G is P_{n-k}-contractible.

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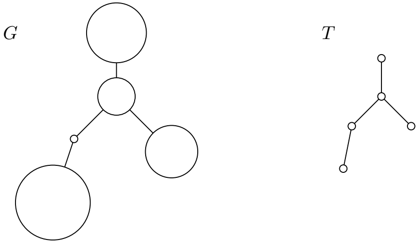

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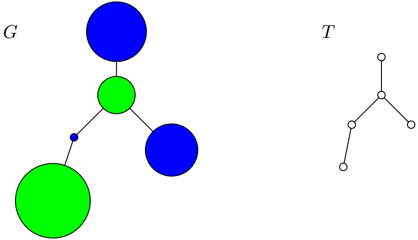

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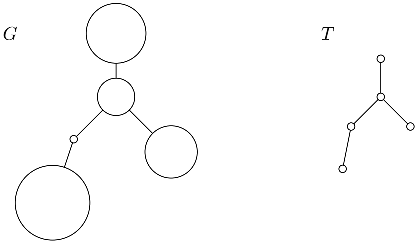
Again, yes!

One more thing about contractions



Observation

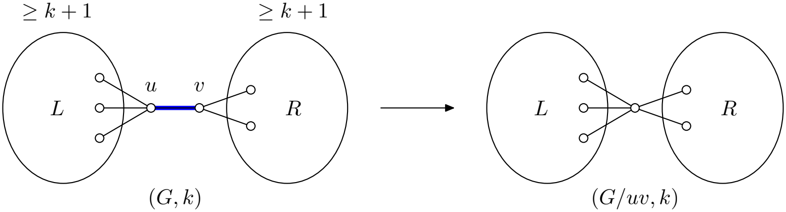
If G is k-contractible to T, then any T-witness structure satisfies:

- *every witness set has size $\leq k+1$;*
- *at most k big witness sets;*
- *at most $2k$ vertices contained in big witness sets.*

Contracting a graph to a path

Linear vertex kernel for Path Contraction

Reduction rule:



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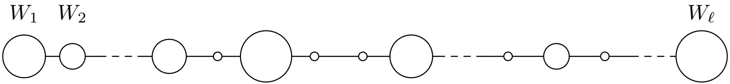
Proof. Suppose G is k-contractible to P_ℓ.



Note: $n \leq \ell + k$.

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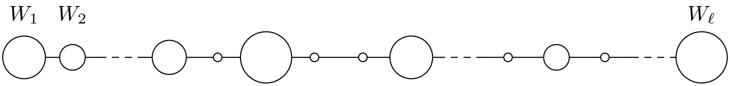

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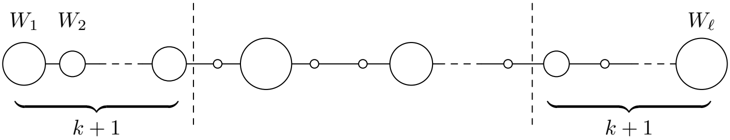

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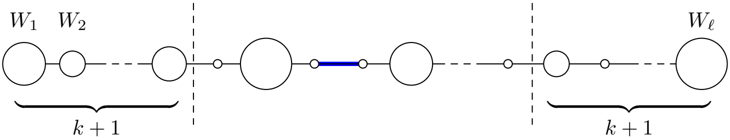

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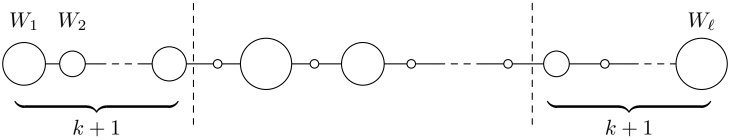

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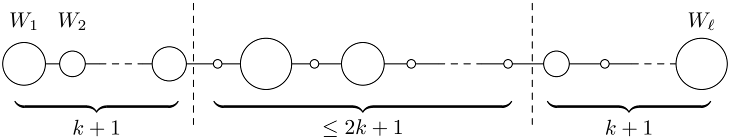

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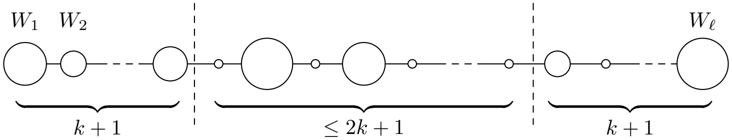

Note: $n \leq \ell + k$. This, together with $\ell \leq 4k+3$, yields the result.

Fast FPT algorithm for Path Contraction

Theorem

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Corollary

Path Contraction *can be solved in* $32^{k+o(k)} + n^{O(1)}$ *time.*

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Theorem (Thomassé, 2009)

FEEDBACK VERTEX SET *has a kernel with* $O(k^2)$ *vertices.*

No polynomial kernel for TREE CONTRACTION...

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...but still a fast FPT algorithm

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TREE CONTRACTION *has no polynomial kernel, unless* $NP \subseteq coNP/poly$.

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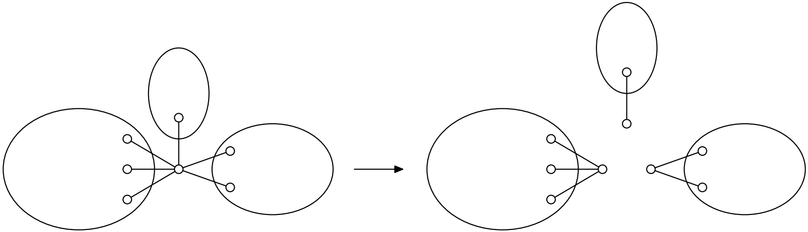
TREE CONTRACTION *can be solved in* $4.98^k \cdot n^{O(1)}$ *time.*

We may assume G to be 2-connected

Lemma

A graph is k-contractible to a tree if and only if each of its 2-connected components can be contracted to a tree, using at most k edge contractions in total.



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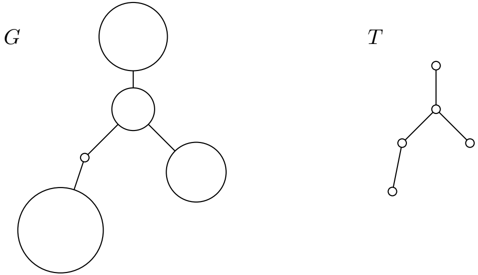

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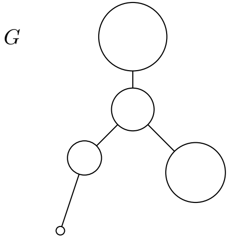

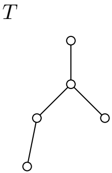

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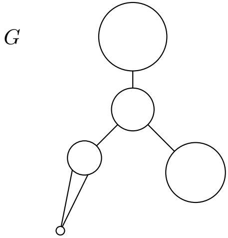
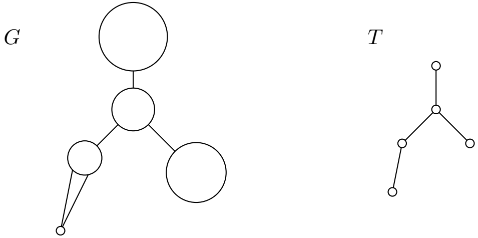


Compatible 2-colorings

Definition

A 2-coloring of G is *compatible* with a T-witness structure if it colors

- each big witness set monochromatically, and
- each pair of adjacent big witness sets differently.



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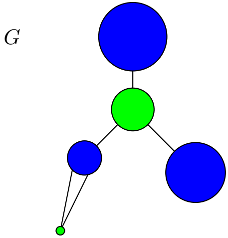

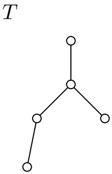

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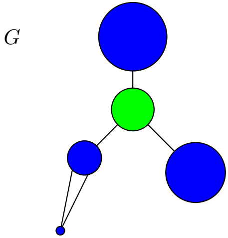

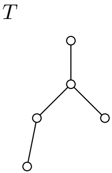

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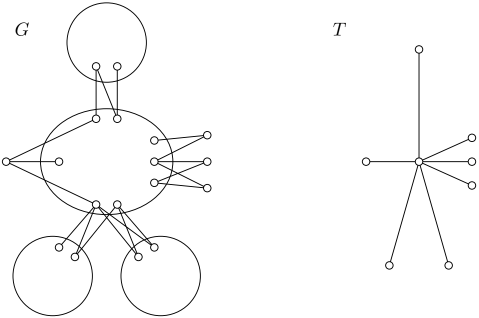

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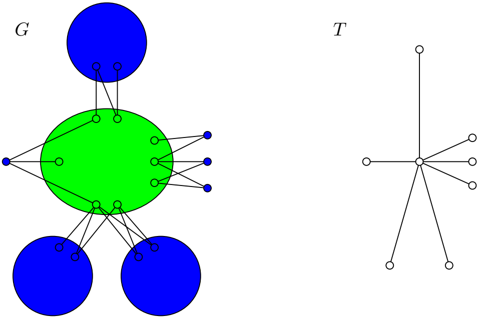

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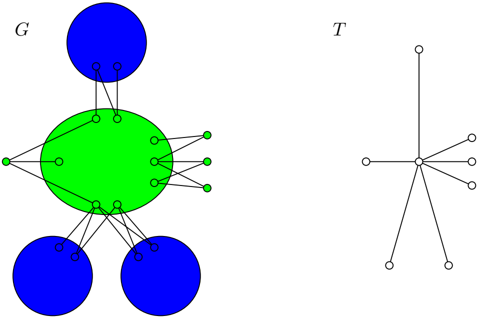

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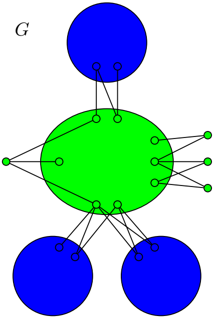

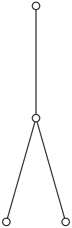

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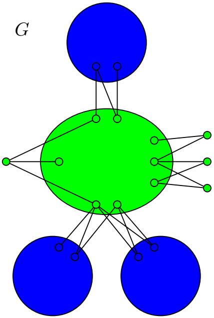

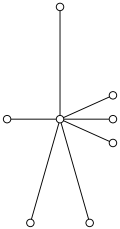

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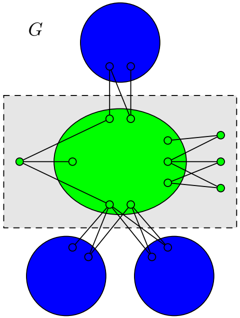

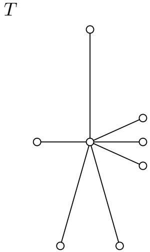

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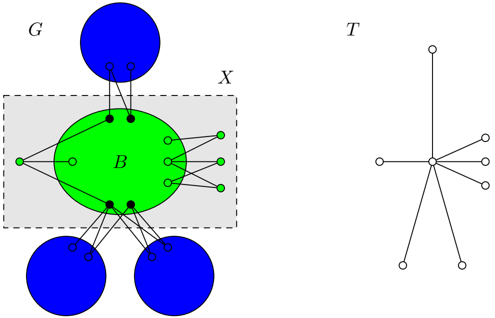
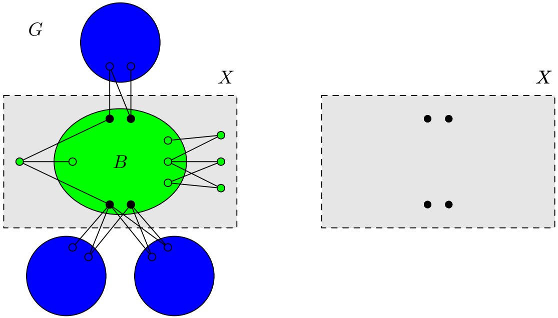


X = monochromatic component of a compatible 2-coloring φ

- X contains at most 1 big T-witness set B
- B is a connected vertex cover of $G[X]$
- B contains all black vertices

From compatible 2-coloring to T-witness structure



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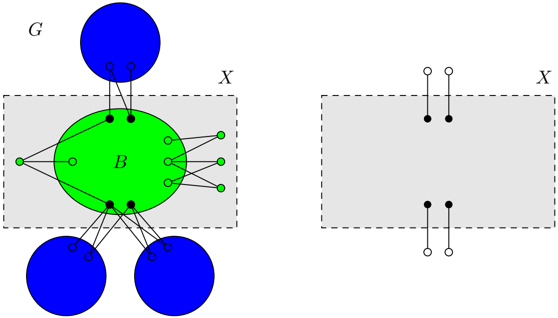

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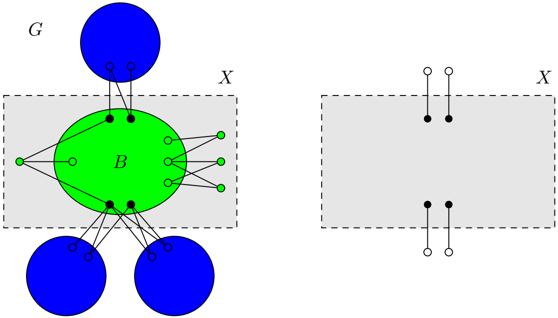

Theorem (Binkele-Raible & Fernau, 2010)

Given a graph G and an integer p, a connected vertex cover of G of size at most p can be found in $2.4882^p \cdot n^{O(1)}$ time, if one exists.

Limiting the number of 2-colorings we have to check

Recall that

- any "good" T-witness structure has at most $2k$ vertices in big witness sets
- compatibility of a 2-coloring depends only on the colors of the vertices in big witness sets

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There is a deterministic algorithm that constructs an $(n, 2k)$-universal set $\mathcal{F}$ of size $4^{k+o(k)} \log n$ in linear time.

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To wrap up

Bipartite Contraction

- fixed-parameter tractable (Heggernes, v. 't H., Lokshtanov & Paul)
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Π-CONTRACTION

- pick your favorite graph class Π, and have fun!

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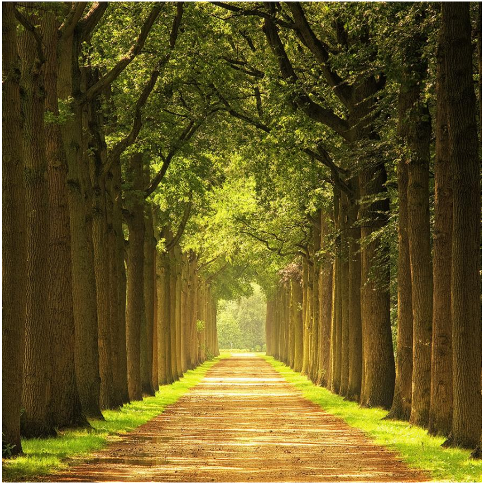
More contractions coming up in the next talk!

That's all

Dank u wel!

Danke!

Takk!



Pim van 't Hof
`http://folk.uib.no/pho042`