# A new characterization of $P_6$-free graphs

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**Abstract.** We study $P_6$-free graphs, i.e., graphs that do not contain an induced path on six vertices. Our main result is a new characterization of this graph class: a graph $G$ is $P_6$-free if and only if each connected induced subgraph of $G$ on more than one vertex contains a dominating induced cycle on six vertices or a dominating (not necessarily induced) complete bipartite subgraph. This characterization is minimal in the sense that there exists an infinite family of $P_6$-free graphs for which a smallest connected dominating subgraph is a (not induced) complete bipartite graph. Our characterization of $P_6$-free graphs strengthens results of Liu and Zhou, and of Liu, Peng and Zhao. Our proof has the extra advantage of being constructive: we present an algorithm that finds such a dominating subgraph of a connected $P_6$-free graph in polynomial time. This enables us to solve the Hypergraph 2-Colorability problem in polynomial time for the class of hypergraphs with $P_6$-free incidence graphs.

## 1 Introduction

All graphs in this paper are undirected, finite, and simple, i.e., without loops and multiple edges. Furthermore, unless specifically stated otherwise, all graphs are non-trivial, i.e., contain at least two vertices. For undefined terminology we refer to [8]. Let $G = (V, E)$ be a graph. For a subset $U \subseteq V$ we denote by $G[U]$ the subgraph of $G$ induced by $U$. A subset $S \subseteq V$ is called a *clique* if $G[S]$ is a complete graph. A set $U \subseteq V$ *dominates* a set $U' \subseteq V$ if any vertex $v \in U'$ either lies in $U$ or has a neighbor in $U$. We also say that $U$ *dominates* $G[U']$. A subgraph $H$ of $G$ is a *dominating subgraph* of $G$ if $V(H)$ dominates $G$. We write $P_k, C_k, K_k$ to denote the path, cycle and complete graph on $k$ vertices, respectively.

A graph $G$ is called $H$*-free* for some graph $H$ if $G$ does not contain an induced subgraph isomorphic to $H$. For any family $\mathcal{F}$ of graphs, let $Forb(\mathcal{F})$ denote the class of graphs that are $F$-free for every $F \in \mathcal{F}$. We consider the class $Forb(\{P_t\})$ of graphs that do not contain an induced path on $t$ vertices. Note that $Forb(\{P_2\})$ is the class of graphs without any edge and $Forb(\{P_3\})$ is the class of graphs all components of which are complete graphs.

* This work has been supported by EPSRC (EP/D053633/1).

† An extended abstract of this paper has been accepted for COCOON 2008.

The class of $P_4$-free graphs (or cographs) has been studied extensively (cf. [5]). The following characterization of $Forb(\{P_4, C_4\})$, i.e., the class of $C_4$-free cographs, is due to Wolk [19, 20] (see also Theorem 11.3.4 in [5]).

**Theorem 1 ([19, 20]).** *A graph $G$ is $P_4$-free and $C_4$-free if and only if each connected induced subgraph of $G$ contains a dominating vertex.*

We can slightly modify this theorem to obtain a characterization of $P_4$-free graphs.

**Theorem 2.** *A graph $G$ is $P_4$-free if and only if each connected induced subgraph of $G$ contains a dominating induced $C_4$ or a dominating vertex.*

Since this theorem can be proven using similar (but much easier) arguments as in the proof of our main result, its proof is omitted here.

The following characterization of $P_5$-free graphs is due to Liu and Zhou [14].

**Theorem 3 ([14]).** *A graph $G$ is $P_5$-free if and only if each connected induced subgraph of $G$ contains a dominating induced $C_5$ or a dominating clique.*

A graph $G$ is called *triangle extended complete bipartite (TECB)* if it is a complete bipartite graph or if it can be obtained from a complete bipartite graph $F$ by adding some extra vertices $w_1, \ldots, w_r$ and edges $w_iu, w_iv$ for $1 \leq i \leq r$ to exactly one edge $uv$ of $F$ (see Figure 1 for an example).

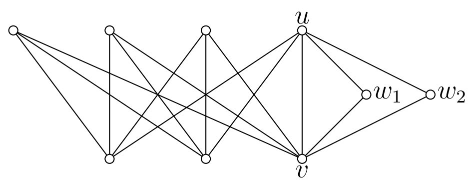


**Fig. 1.** An example of a TECB graph.

The following characterization of $P_6$-free graphs is due to Liu, Peng and Zhao [15].

**Theorem 4 ([15]).** *A graph $G$ is $P_6$-free if and only if each connected induced subgraph of $G$ contains a dominating induced $C_6$ or a dominating (not necessarily induced) TECB graph.*

If we consider graphs that are not only $P_6$-free but also triangle-free, then we have one of the main results in [14].

**Theorem 5 ([14]).** *A triangle-free graph $G$ is $P_6$-free if and only if each connected induced subgraph of $G$ contains a dominating induced $C_6$ or a dominating (not necessarily induced) complete bipartite graph.*

A characterization of $Forb(\{P_t\})$ for $t \geq 7$ is given in [1]: $Forb(\{P_t\})$ is the class of graphs for which each connected induced subgraph has a dominating subgraph of diameter at most $t-4$.

**Our results**

Section 3 contains our main result.

**Theorem 6.** *A graph $G$ is $P_6$-free if and only if each connected induced subgraph of $G$ contains a dominating induced $C_6$ or a dominating (not necessarily induced) complete bipartite graph. Moreover, we can find such a dominating subgraph in polynomial time.*

This theorem strengthens Theorem 4 and Theorem 5 in two different ways. Firstly, Theorem 6 shows that we may omit the restriction "triangle-free" in Theorem 5 and that we may replace the class of TECB graphs by its proper subclass of complete bipartite graphs in in Theorem 4. Secondly, in contrast to the proofs of Theorem 4 and Theorem 5, the proof of Theorem 6 is constructive: we provide a (polynomial time) algorithm for finding the desired dominating subgraph. Note that we cannot use some brute force approach to obtain such a polynomial time algorithm, since a dominating complete bipartite graph might have arbitrarily large size.

In Section 3, we also show that the characterization in Theorem 6 is minimal in the sense that there exists an infinite family of $P_6$-free graphs for which a smallest connected dominating subgraph is a (not induced) complete bipartite graph. We would like to mention that the algorithm used to prove Theorem 6 also works for an arbitrary (not necessarily $P_6$-free) graph $G$: in that case the algorithm either finds a dominating subgraph as described in Theorem 6 or finds an induced $P_6$ in $G$. Furthermore, we can easily modify our algorithm so that it finds a dominating induced $C_5$ or a dominating clique of a $P_5$-free graph in polynomial time. This yields a constructive proof of Theorem 3 and generalizes the algorithm by Cozzens and Kelleher [7] that finds a dominating clique of a connected graph without an induced $P_5$ or $C_5$. We end Section 3 by characterizing the class of graphs for which each connected induced subgraph has a dominating induced $C_6$ or a dominating *induced* complete bipartite subgraph (again by giving a constructive proof). This class consists of graphs that, apart from $P_6$, have exactly one more forbidden induced subgraph. This generalizes a result in [2].

As an application of our main result, we consider the Hypergraph 2-Colorability problem in Section 4. It is well-known that this problem is NP-complete in general (cf. [10]). We prove that for the class of hypergraphs with $P_6$-free incidence graphs the problem becomes polynomially solvable. Moreover, we show that for any 2-colorable hypergraph $H$ with a $P_6$-free incidence graph, we can find a 2-coloring of $H$ in polynomial time.

Section 5 contains the conclusions, discusses a number of related results in the literature and mentions open problems.

## 2 Preliminaries

We use the following terminology throughout the paper for a graph $G = (V, E)$. We say that an order $\pi = x_1, \ldots, x_{|V|}$ of $V$ is *connected* if $G_i := G[\{x_1, \ldots, x_i\}]$

is connected for $i = 1, \ldots, |V|$. Let $w \in V$ and $D \subseteq V$. Then $N_G(w)$ denotes the set of neighbors of $w$ in $G$. We write $N_D(w) := N_G(w) \cap D$ and $N_G(D) := \cup_{u \in D} N_G(u) \backslash D$. If no confusion is possible, we write $N(w)$ (respectively $N(D)$) instead of $N_G(w)$ (respectively $N_G(D)$). A vertex $v' \in V \backslash D$ is called a *$D$-private neighbor* (or simply private neighbor if no confusion is possible) of a vertex $v \in D$ if $N_D(v') = \{v\}$.

Let $u, v$ be a pair of adjacent vertices in a dominating set $D$ of a graph $G$ such that $\{u, v\}$ dominates $D$. We call a dominating set $D' \subseteq D$ of $G$ a *minimizer of $D$ for $uv$* if $\{u, v\} \subseteq D'$ and each vertex of $D' \backslash \{u, v\}$ has a $D'$-private neighbor in $G$. We can obtain such a minimizer $D'$ from $D$ in polynomial time by repeatedly removing vertices without private neighbor from $D \backslash \{u, v\}$. This can be seen as follows. It is clear that $D'$ dominates all vertices in $V(G) \backslash D$, since we only remove a vertex from $D \backslash \{u, v\}$ if all its neighbors outside $D$ are dominated by remaining vertices in $D$. Moreover, since $\{u, v\}$ dominates $D$, all vertices removed from $D \backslash \{u, v\}$ are dominated by $\{u, v\}$. Note that the fact that $u$ and $v$ are adjacent means that the graph $G[D']$ is connected. We point out that $D$ may have several minimizers for the same edge $uv$ depending on the order in which its vertices are considered.

**Example.** Consider the graph $G$ and its connected dominating set $D$ in the left-hand side of Figure 2. All private neighbors are colored black. The set $D'$ in the right-hand side is a minimizer of $D$ for $uv$ obtained by removing $w_4$ from $D$. Note that $u$ does not have a $D'$-private neighbor but $v$ does. Instead of removing $w_4$ we could also have chosen to remove $w_2$ first, since $w_2$ does not have a $D$-private neighbor. Let $D^1 := D \backslash \{w_2\}$. Since $w_3$ does not have a $D^1$-private neighbor, we can remove $w_3$ from $D^1$. The resulting set $D^2 := D^1 \backslash \{w_3\}$ is a minimizer of $D$ for $uv$ in which every vertex of $D^2$ (including $u$) has a $D^2$-private neighbor.

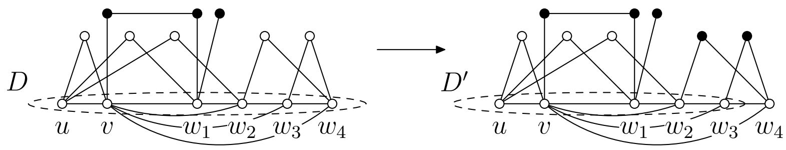


**Fig. 2.** A dominating set $D$ and a minimizer $D'$ of $D$ for $uv$.

## 3 Finding connected dominating subgraphs in $P_6$-free graphs

Let $G$ be a connected $P_6$-free graph. We say that $D$ is a *type 1 dominating set* of $G$ if $D$ dominates $G$ and $G[D]$ is an induced $C_6$. We say that $D$ is a *type 2 dominating set* of $G$ *defined by* $A(D)$ and $B(D)$ if $D$ dominates $G$ and $G[D]$ contains a spanning complete bipartite subgraph with partition classes $A(D)$ and $B(D)$.

**Theorem 7.** *If $G$ is a connected $P_6$-free graph, then we can find a type 1 or type 2 dominating set of $G$ in polynomial time.*

*Proof.* Let $G = (V, E)$ be a connected $P_6$-free graph with connected order $\pi = x_1, \ldots, x_{|V|}$. Recall that we write $G_i := G[\{x_1, \ldots, x_i\}]$, and note that $G_i$ is connected and $P_6$-free for every $i$. For every $2 \leq i \leq n$ we want to find a type 1 or type 2 dominating set $D_i$ of $G_i$. Let $D_2 := \{x_1, x_2\}$. Suppose $i \geq 3$. Assume $D_{i-1}$ is a type 1 or type 2 dominating set of $G_{i-1}$. We show how we can use $D_{i-1}$ to find $D_i$ in polynomial time. Since the total number of iterations is $|V|$, we then find a desired dominating subgraph of $G_{|V|} = G$ in polynomial time. We write $x := x_i$. If $x \in N(D_{i-1})$, then we set $D_i := D_{i-1}$. Suppose otherwise. Since $\pi$ is connected, $G_i$ contains a vertex $y$ (not in $D_{i-1}$) adjacent to $x$.

**Case 1.** $D_{i-1}$ **is a type 1 dominating set of** $G_{i-1}$**.**

We write $G[D_{i-1}] = c_1c_2c_3c_4c_5c_6c_1$. We claim that $D := N_{D_{i-1}}(y) \cup \{x, y\}$ dominates $G_i$, which means that $D_i := D$ is a type 2 dominating set of $G_i$ defined by $A(D_i) := \{y\}$ and $B(D_i) := \{x\} \cup N_{D_{i-1}}(y)$. Suppose $D$ does not dominate $G_i$, and let $z \in V(G_i)$ be a vertex not dominated by $D$. Since $D_{i-1}$ dominates $G_{i-1}$, we may without loss of generality assume that $yc_1 \in E(G_i)$.

Suppose $yc_4 \in E(G_i)$. Note that $z$ is dominated by $G_{i-1}$. Without loss of generality, assume $z$ is adjacent to $c_2$. Consequently, $y$ is not adjacent to $c_2$. Since $z$ is not adjacent to any neighbor of $y$ and the path $zc_2c_1yc_4c_5$ cannot be induced in $G_i$, either $z$ or $y$ must be adjacent to $c_5$. If $zc_5 \in E(G_i)$, then $xyc_4c_5zc_2$ is an induced $P_6$ in $G_i$. Hence $zc_5 \notin E(G_i)$ and $yc_5 \in E(G_i)$. In case $zc_6 \in E(G_i)$ we obtain an induced path $xyc_5c_6zc_2$ on six vertices, and in case $zc_6 \notin E(G_i)$ we obtain an induced path $zc_2c_1c_6c_5c_4$. We conclude $yc_4 \notin E(G_i)$.

Suppose $y$ is not adjacent to any vertex in $\{c_3, c_5\}$. Since $G_i$ is $P_6$-free and $xyc_1c_2c_3c_4$ is a $P_6$ in $G_i$, $y$ must be adjacent to $c_2$. But then $xyc_2c_3c_4c_5$ is an induced $P_6$ in $G_i$, a contradiction. Hence $y$ is adjacent to at least one vertex in $\{c_3, c_5\}$, say $yc_5 \in E(G_i)$. By symmetry (using $c_5, c_2$ instead of $c_1, c_4$) we find $yc_2 \notin E(G_i)$.

Suppose $z$ is adjacent to $c_2$. The path $zc_2c_1yc_5c_4$ on six vertices and the $P_6$-freeness of $G_i$ imply $zc_4 \in E(G_i)$. But then $c_2zc_4c_5yx$ is an induced $P_6$. Hence $zc_2 \notin E(G_i)$. Also $zc_4 \notin E(G_i)$ as otherwise $zc_4c_5yc_1c_2$ would be an induced $P_6$, and $zc_3 \notin E(G_i)$ as otherwise $zc_3c_2c_1yx$ would be an induced $P_6$. Then $z$ must be adjacent to $c_6$ yielding an induced path $zc_6c_1c_2c_3c_4$ on six vertices. Hence we may choose $D_i := D$.

**Case 2.** $D_{i-1}$ **is a type 2 dominating set of** $G_{i-1}$**.**

Since $D_{i-1}$ dominates $G_{i-1}$, we may assume that $y$ is adjacent to some vertex $a \in A(D_{i-1})$. Let $b \in B(D_{i-1})$. Let $D$ be a minimizer of $D_{i-1} \cup \{y\}$ for $ab$ (note that $\{a, b\}$ dominates $D_{i-1} \cup \{y\}$). By definition, $D$ dominates $G_i$. Also, $G[D]$ contains a spanning (not necessarily complete) bipartite graph with partition classes $A \subseteq A(D_{i-1}), B \subseteq B(D_{i-1}) \cup \{y\}$. Note that we have $y \in D$, because $x$ is not adjacent to $D_{i-1}$ and therefore is a $D$-private neighbor of $y$. Since $y$ might not have any neighbors in $B$ but does have a neighbor (vertex $a$) in $A$, we chose $y \in B$.

*Claim 1. If $G[D]$ contains an induced $P_4$ starting in $y$ and ending in some $r \in A$, then we can find a type 1 or a type 2 dominating set $D_i$ of $G_i$ in polynomial time.*

We prove Claim 1 as follows. Suppose $ypqr$ is an induced path in $G[D]$ with $r \in A$. Since $D$ is a minimizer of $D_{i-1} \cup \{y\}$ for $ab$ and $r \in D \backslash \{a, b\}$, $r$ has a $D$-private neighbor $s$ by definition. Since $xypqrs$ is a path on six vertices and $x \notin N(D_{i-1})$ holds, $x$ must be adjacent to $s$. We first show that $D^1 := N_D(y) \cup \{x, y, q, r, s\}$ dominates $G_i$. See Figure 3 for an illustration of the graph $G[D^1]$. Suppose $D^1$

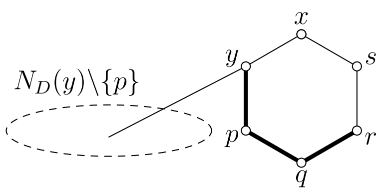


**Fig. 3.** The graph $G[D^1]$.

does not dominate $G$. Then there exists a vertex $z \in N(D) \backslash N(D^1)$. Note that $G[(D\backslash\{y\}) \cup \{z\}]$ is connected because the edge $ab$ makes $D\backslash\{y\}$ connected and $\{a, b\}$ dominates $D$. Let $P$ be a shortest path in $G[(D\backslash\{y\}) \cup \{z\}]$ from $z$ to a vertex $p_1 \in N_D(y)$ (possibly $p_1 = p$). Since $z \notin N(D^1)$ and $p_1 \in D^1$, we have $|V(P)| \geq 3$. This means that $Pyxs$ is an induced path on at least six vertices, unless $r \in V(P)$ (since $r$ is adjacent to $s$). However, if $r \in V(P)$, then the subpath $z\overrightarrow{P}r$ of $P$ from $z$ to $r$ has at least three vertices (because $z \notin N(D^1)$). This means that $z\overrightarrow{P}rsxy$ contains an induced $P_6$, a contradiction. Hence $D^1$ dominates $G_i$.

To find a type 1 or type 2 dominating set $D_i$ of $G_i$, we transform $D^1$ into $D_i$ as follows. Suppose $q$ has a $D^1$-private neighbor $q'$. Then $q'qpyxs$ is an induced $P_6$ in $G_i$, a contradiction. Hence $q$ has no $D^1$-private neighbor and the set $D^2 := D^1 \backslash \{q\}$ still dominates $G_i$. Similarly, $r$ has no $D^2$-private neighbor $r'$, since otherwise $r'rsxyp$ would be an induced $P_6$ in $G_i$. So the set $D^3 := D^2 \backslash \{r\}$ also dominates $G_i$. Now suppose $s$ does not have a $D^3$-private neighbor. Then the set $D^3 \backslash \{s\}$ dominates $G_i$. In that case, we find a type 2 dominating set $D_i$ of $G_i$ defined by $A(D_i) := \{y\}$ and $B(D_i) := N_D(y) \cup \{x\}$. Assume that $s$ has a $D^3$-private neighbor $s'$ in $G_i$. Let $D^4 := D^3 \cup \{s'\}$.

Suppose $N_D(y)\backslash\{p\}$ contains a vertex $p_2$ that has a $D^4$-private neighbor $p_2'$. Then $p_2'p_2yxss'$ is an induced $P_6$, contradicting the $P_6$-freeness of $G_i$. Hence we can remove all vertices of $N_D(y)\backslash\{p\}$ from $D^4$, and the resulting set $D^5 := \{p, y, x, s, s'\}$ still dominates $G_i$. We claim that $D^6 := D^5 \cup \{q\}$ is a type 1 dominating set of $G_i$. Clearly, $D^6$ dominates $G_i$, since $D^5 \subseteq D^6$. Since $qpyxss'$

is a $P_6$ and $qpyxs$ is induced, $q$ must be adjacent to $s'$. Hence $D^6$ is a type 1 dominating set of $G_i$, and we choose $D_i := D^6$. This proves Claim 1.

Let $A_1 := N_A(y)$ and $A_2 := A \backslash A_1$. Let $B_1 := N_B(y)$ and $B_2 := B \backslash (B_1 \cup \{y\})$. Since $a \in A_1$, we have $A_1 \neq \emptyset$. If $A_2 = \emptyset$, then we define a type 2 dominating set $D_i$ of $G_i$ by $A(D_i) := A$ and $B(D_i) := B$. Suppose $A_2 \neq \emptyset$. Note $|B| \geq 2$, because $\{b, y\} \subseteq B$. If $B_2 = \emptyset$, then we define $D_i$ by $A(D_i) := A \cup \{y\}$ and $B(D_i) := B_1 = B \backslash \{y\}$. Suppose $B_2 \neq \emptyset$. If $G[A_1 \cup A_2]$ contains a spanning complete bipartite graph with partition classes $A_1$ and $A_2$, we define $D_i$ by $A(D_i) := A_1$ and $B(D_i) := A_2 \cup B$. Hence we may assume that there exist two non-adjacent vertices $a_1 \in A_1$ and $a_2 \in A_2$. Let $b^* \in B_2$. Then $ya_1b^*a_2$ is an induced $P_4$ starting in $y$ and ending in a vertex of $A$. By Claim 1, we can find a type 1 or type 2 dominating set $D_i$ of $G_i$ in polynomial time. This finishes the proof of Theorem 7. $\square$

We will now prove our main theorem.

**Theorem 6.** *A graph $G$ is $P_6$-free if and only if each connected induced subgraph of $G$ contains a dominating induced $C_6$ or a dominating (not necessarily induced) complete bipartite graph. Moreover, we can find such a dominating subgraph in polynomial time.*

*Proof.* Let $G$ be a graph. Suppose $G$ is not $P_6$-free. Then $G$ contains an induced $P_6$ which contains neither a dominating induced $C_6$ nor a dominating complete bipartite graph. Suppose $G$ is $P_6$-free. Let $H$ be a connected induced subgraph of $G$. Then $H$ is $P_6$-free as well. We apply Theorem 7 to $H$. $\square$

The characterization in Theorem 6 is minimal due to the existence of the following family $\mathcal{F}$ of $P_6$-free graphs. For each $i \geq 2$, let $F_i \in \mathcal{F}$ be the graph obtained from a complete bipartite subgraph with partition classes $X_i = \{x_1, \ldots, x_i\}$ and $Y_i = \{y_1, \ldots, y_i\}$ by adding the edge $x_1x_2$ as well as for each $h = 1, \ldots, i$ a new vertex $x'_h$ only adjacent to $x_h$ and a new vertex $y'_h$ only adjacent to $y_h$ (see Figure 4 for the graph $F_3$).

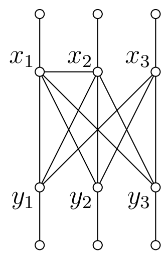


**Fig. 4.** The graph $F_3$.

Note that each $F_i$ is $P_6$-free and that the smallest connected dominating subgraph of $F_i$ is $F_i[X_i \cup Y_i]$, which contains a spanning complete bipartite subgraph.

Also note that none of the graphs $F_i$ contain a dominating *induced* complete bipartite subgraph due to the edge $x_1x_2$.

We conclude this section by characterizing the class of graphs for which each connected induced subgraph contains a dominating induced $C_6$ or a dominating *induced* complete bipartite subgraph. Again, we will show how to find these dominating induced subgraphs in polynomial time. Let $C_3^L$ denote the graph obtained from the cycle $c_1c_2c_3c_1$ by adding three new vertices $b_1, b_2, b_3$ and three new edges $c_1b_1, c_2b_2, c_3b_3$ (see Figure 5).

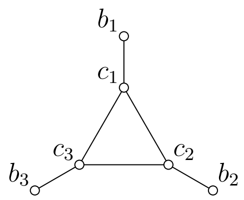


**Fig. 5.** The graph $C_3^L$.

**Theorem 8.** *If $G$ is a connected graph in Forb$(\{C_3^L, P_6\})$, then we can find a dominating induced $C_6$ or a dominating induced complete bipartite subgraph of $G$ in polynomial time.*

*Proof.* Let $G = (V, E)$ be a connected graph in *Forb*$(\{C_3^L, P_6\})$ with connected order $\pi = x_1, \ldots, x_{|V|}$. Recall that we write $G_i := G[\{x_1, \ldots, x_i\}]$, and note that $G_i \in$ *Forb*$(\{C_3^L, P_6\})$ for every $i$. For every $2 \le i \le n$ we want to find a dominating set $D_i$ of $G_i$ that either induces a $C_6$ or a complete bipartite subgraph in $G_i$. Let $D_2 := \{x_1, x_2\}$. Suppose $i \ge 3$. Assume $D_{i-1}$ induces a dominating $C_6$ or a dominating complete bipartite subgraph in $G_{i-1}$. We show how we can use $D_{i-1}$ to find $D_i$ in polynomial time. Since the total number of iterations is $|V|$, we find a desired dominating subgraph of $G_{|V|} = G$ in polynomial time. We write $x := x_i$. If $x \in N(D_{i-1})$, then we set $D_i := D_{i-1}$. Suppose otherwise. Since $\pi$ is connected, $G_i$ contains a vertex $y$ (not in $D_{i-1}$) adjacent to $x$. We first prove a useful claim.

*Claim 1. If $N_{D_{i-1}}(y) \cup \{x, y\}$ dominates $G_i$, then we can find a dominating induced $C_6$ or a dominating induced complete bipartite subgraph of $G$ in polynomial time.*

We prove Claim 1 as follows. Suppose $D^* := N_{D_{i-1}}(y) \cup \{x, y\}$ dominates $G_i$. We check whether $G[D^*]$ is complete bipartite. If so, then we choose $D_i := D^*$ and we are done. Otherwise $y$ has a neighbor $u$ in $D_{i-1}$ with $N_{D^*}(u) \backslash \{y\} \neq \emptyset$. If $u$ has no $D^*$-private neighbor, then we remove $u$ from $D^*$ and perform the same check in the smaller set $D^* \backslash \{u\}$. Let $u'$ be a $D^*$-private neighbor of $u$ in $G_i$. Let $v \in N_{D^*}(u) \backslash \{y\}$. Then $u'$ is adjacent to any $D^*$-private neighbor

$v'$ of $v$, as otherwise $G[\{u,v,y,u',v',x\}]$ is isomorphic to $C_3^L$. So we find that $D^1 := (D^*\backslash N_{D^*}(u)) \cup \{y,u'\}$ dominates $G_i$. If $u'$ does not have a $D^1$-private neighbor, then we remove $u'$ from $D^1$, check if $y$ is adjacent to two neighbors in the smaller set $D^1\backslash\{u'\}$ and repeat the above procedure. Let $u''$ be a $D^1$-private neighbor of $u'$. Suppose $N_{D^1}(y) = \{x,u\}$. Then $D^1 = \{x,y,u,u'\}$. If $x$ does not have a $D^1$-private neighbor, then we choose $D_i := \{y,u,u'\}$. If $x$ has a $D^1$-private neighbor $x'$, then the $P_6$-freeness of $G_i$ implies that $x'$ is adjacent to $u''$, and we choose $D_i := \{x',x,y,u,u',u''\}$.

Suppose $N_{D^1}(y)\backslash\{x,u\} \neq \emptyset$, say $y$ is adjacent to some vertex $t \in D^1\backslash\{x,u\}$. If $t$ does not have a $D^1$-private neighbor, then we remove $t$ from $D^1$ and check if $y$ is adjacent to some vertex in the smaller set $D^1\backslash\{x,u,t\}$. Let $t'$ be a $D^1$-private neighbor of $t$. Then the path $u''u'uytt'$ is an induced $P_6$ of $G_i$, unless $u''$ is adjacent to $t'$. However, in that case $xyuu'u''t'$ is an induced $P_6$. This finishes the proof of Claim 1.

**Case 1. $D_{i-1}$ induces a dominating $C_6$ in $G_{i-1}$.**

Since $D_{i-1}$ is a type 1 dominating set of $G_{i-1}$, we know from the corresponding Case 1 in the proof of Theorem 7 that $D := N_{D_{i-1}}(y) \cup \{x,y\}$ dominates $G_i$. By Claim 1, we can find a dominating induced $C_6$ or a dominating induced complete bipartite subgraph of $G$ in polynomial time.

**Case 2. $D_{i-1}$ induces a dominating complete bipartite subgraph in $G_{i-1}$.**

Let $A(D_{i-1})$ and $B(D_{i-1})$ denote the partition classes of $D_{i-1}$. Note that both $A(D_{i-1})$ and $B(D_{i-1})$ are independent sets. Since $D_{i-1}$ dominates $G_{i-1}$, we may without loss of generality assume that $y$ is adjacent to some vertex $a \in A(D_{i-1})$. Let $b \in B(D_{i-1})$. Let $D$ be a minimizer of $D_{i-1} \cup \{y\}$ for $ab$. By definition, $D$ dominates $G_i$. Also, $G[D]$ contains a spanning (not necessarily complete) bipartite graph with partition classes $A \subseteq A(D_{i-1})$ and $B \subseteq B(D_{i-1}) \cup \{y\}$. Note that $y \in D$, because $x$ is not adjacent to $D_{i-1}$ and therefore is a $D$-private neighbor of $y$, and consequently, $y \in B$ because $y$ is adjacent to $a \in A$ (and $y$ might not have any neighbors in $B$). Let $A_1 := N_A(y)$ and $A_2 := A\backslash A_1$. Let $B_1 := N_B(y)$ and $B_2 := B\backslash(B_1 \cup \{y\})$. Since $a \in A_1$, we have $A_1 \neq \emptyset$.

Suppose $G[D]$ contains an induced $P_4$ starting in $y$ and ending in a vertex in $A$. Just as in the proof of Theorem 7 we can obtain (in polynomial time) a dominating $C_6$ of $G_i$ or else we find that $N_D(y) \cup \{x,y\}$, and consequently, $N_{D_{i-1}}(y) \cup \{x,y\}$ dominates $G_i$. In the first case, we choose $D_i$ to be the obtained dominating induced $C_6$. In the second case, we can find a dominating induced $C_6$ or a dominating induced complete bipartite subgraph of $G$ in polynomial time by Claim 1. So we may assume that $G[D]$ does not contain such an induced $P_4$. This means that at least one of the sets $A_2, B_2$ is empty, as otherwise we find an induced path $yab_2a_2$ for any $a_2 \in A_2$ and $b_2 \in B_2$. We may without loss of generality assume that $A_2 = \emptyset$. (Otherwise, in case $B_2 = \emptyset$, we obtain $B = B_1$, which means that $y$ is adjacent to $b$ as well, so we can reverse the role of $A$ and $B$.) If $B_2 = \emptyset$, then we find that $A_1 \cup B_1 \cup \{y\} \subset N_{D_{i-1}}(y) \cup \{x,y\}$ dominates $G_i$, and we are done as a result of Claim 1. So $B_2 \neq \emptyset$. Let $b_2 \in B_2$.

We claim that $D^2 := A_1 \cup B_2 \cup \{x, y\}$ dominates $G_i$. Suppose otherwise. Then there exists a vertex $b_1'$ adjacent to some vertex $b_1 \in B_1$ but not adjacent to $D^2$. Then $G[\{y, a, b_1, x, b_2, b_1'\}]$ is isomorphic to $C_3^L$, a contradiction. Hence $D^2$ dominates $G_i$. If $x$ does not have a $D^2$-private neighbor, then we can choose $D_i := D^2 \backslash \{x\}$, since $G[D_i]$ is a complete bipartite graph with partition classes $A_1$ and $B_2 \cup \{y\}$. Suppose $x$ has a $D^2$-private neighbor $x'$. If $b_2$ does not have a $D^2$-private neighbor, then we remove $b_2$ from $D^2$, and check whether $B_2$ contains another vertex (if not we are done, i.e., we can choose $D_i := A_1 \cup \{x, y\}$, since $G[D_i]$ is a complete bipartite graph with partition classes $A_1 \cup \{x\}$ and $\{y\}$). Suppose $b_2$ has a $D^2$-private neighbor $b_2'$. Then the path $x'xyab_2b_2'$ is a path on six vertices, so we must have $x'b_2' \in E$.

We claim that $D^3 := \{x', x, y, a, b_2, b_2'\}$ dominates $G_i$. Suppose otherwise. Then there exists a vertex $c'$ adjacent to some vertex $c$ in $A_1 \cup B_2$ but not adjacent to a vertex in $D^3$. Suppose $c \in A_1$. Then $c'cb_2b_2'x'x$ is an induced $P_6$. Suppose $c \in B_2$. Then $c'cayxx'$ is an induced $P_6$. So $D^3$ dominates $G_i$. Since $D^3$ also induces a $C_6$ in $G_i$, we may choose $D_i := D^3$. This finishes the proof of Theorem 8. ☐

Theorem 9 is an immediate result of Theorem 8 together with the observation that neither the graph $P_6$ nor $C_3^L$ has a dominating induced $C_6$ or a dominating induced complete bipartite subgraph.

**Theorem 9.** *A graph $G$ is in Forb($\{C_3^L, P_6\}$) if and only if each connected induced subgraph of $G$ contains a dominating induced $C_6$ or a dominating induced complete bipartite graph. Moreover, we can find such a dominating subgraph in polynomial time.*

Bacsó, Michalak and Tuza [2] prove (non-constructively) that a graph $G$ is in *Forb*($\{C_3^L, C_6, P_6\}$) if and only if each connected induced subgraph of $G$ contains a dominating induced complete bipartite graph. Note that Theorem 9 immediately implies this result.

## 4 The Hypergraph 2-Colorability problem

A *hypergraph* is a pair $(Q, \mathcal{S})$ consisting of a set $Q = \{q_1, \ldots, q_m\}$ and a set $\mathcal{S} = \{S_1, \ldots, S_n\}$ of subsets of $Q$. With a hypergraph $(Q, \mathcal{S})$ we associate its *incidence graph* $I$, which is a bipartite graph with partition classes $Q$ and $\mathcal{S}$, where for any $q \in Q, S \in \mathcal{S}$ we have $qS \in E(I)$ if and only if $q \in S$. For any $S \in \mathcal{S}$, we write $H - S := (Q, \mathcal{S} \backslash S)$. A 2-*coloring* of a hypergraph $(Q, \mathcal{S})$ is a partition $(Q_1, Q_2)$ of $Q$ such that $Q_1 \cap S_j \neq \emptyset$ and $Q_2 \cap S_j \neq \emptyset$ for $1 \leq j \leq n$.

The Hypergraph 2-Colorability problem asks whether a given hypergraph has a 2-coloring. This is a well-known NP-complete problem (cf. [10]). Let $\mathcal{H}_6$ denote the class of hypergraphs with $P_6$-free incidence graphs.

**Theorem 10.** *The* Hypergraph 2-Colorability *problem restricted to $\mathcal{H}_6$ is polynomially solvable. Moreover, for any 2-colorable hypergraph $H \in \mathcal{H}_6$ we can find a 2-coloring of $H$ in polynomial time.*

*Proof.* Let $H = (Q, \mathcal{S}) \in \mathcal{H}_6$, and let $I$ be the ($P_6$-free) incidence graph of $H$. We assume that $I$ is connected, as otherwise we just proceed component-wise.

*Claim 1. We may without loss of generality assume that $\mathcal{S}$ does not contain two sets $S_i, S_j$ with $S_i \subseteq S_j$.*

We prove Claim 1 as follows. Suppose $S_i, S_j \in \mathcal{S}$ with $S_i \subseteq S_j$. Note that we can check in polynomial time whether such sets $S_i, S_j$ exist. We show that $H$ is 2-colorable if and only if $H - S_j$ is 2-colorable. Clearly, if $H$ is 2-colorable then $H - S_j$ is 2-colorable. Suppose $H - S_j$ is 2-colorable. Let $(Q_1, Q_2)$ be a 2-coloring of $H - S_j$. By definition, $S_i \cap Q_1 \neq \emptyset$ and $S_i \cap Q_2 \neq \emptyset$. Since $S_i \subseteq S_j$, we also have $S_j \cap Q_1 \neq \emptyset$ and $S_j \cap Q_2 \neq \emptyset$, so $(Q_1, Q_2)$ is a 2-coloring of $H$. This proves Claim 1.

By Theorem 6, we can find a type 1 or type 2 dominating set $D$ of $I$ in polynomial time. Since $I$ is bipartite, $G[D]$ is bipartite. Let $A$ and $B$ be the partition classes of $G[D]$. Since $I$ is connected, we may without loss of generality assume $A \subseteq Q$ and $B \subseteq \mathcal{S}$. Let $A' := Q \backslash A$ and $B' := \mathcal{S} \backslash B$. We distinguish two cases.

**Case 1. $D$ is a type 1 dominating set of $I$.**

We write $I[D] = q_1 S_1 q_2 S_2 q_3 S_3 q_1$, so $A = \{q_1, q_2, q_3\}$ and $B = \{S_1, S_2, S_3\}$. Suppose $A' = \emptyset$, so $Q = \{q_1, q_2, q_3\}$. Obviously, $H$ has no 2-coloring. Suppose $A' \neq \emptyset$ and let $q' \in A'$. Since $D$ dominates $I$, $q'$ has a neighbor, say $S_1$, in $B$. If $S_2$ and $S_3$ both have no neighbors in $A'$, then $q' S_1 q_2 S_2 q_3 S_3$ is an induced $P_6$ in $I$, a contradiction. Hence at least one of them, say $S_2$, has a neighbor in $A'$.

We claim that the partition $(Q_1, Q_2)$ of $Q$ with $Q_1 := A' \cup \{q_1\}$ and $Q_2 := \{q_2, q_3\}$ is a 2-coloring of $H$. We have to check that every $S \in \mathcal{S}$ has a neighbor in both $Q_1$ and $Q_2$. Recall that $S_1$ has neighbors $q_1$ and $q_2$ and $S_3$ has neighbors $q_1$ and $q_3$, so $S_1$ and $S_3$ are OK. Since $S_2$ is adjacent to $q_2$ and has a neighbor in $A'$, $S_2$ is also OK. It remains to check the vertices in $B'$. Let $S \in B'$. Since $D$ dominates $I$ and $I$ is bipartite, $S$ has at least one neighbor in $A$. Suppose $S$ has exactly one neighbor, say $q_1$, in $A$. Then $S q_1 S_1 q_2 S_2 q_3$ is an induced $P_6$ in $I$, a contradiction. Hence $S$ has at least two neighbors in $A$. The only problem occurs if $S$ is adjacent to $q_2$ and $q_3$ but not to $q_1$. However, since $S_2$ is adjacent to $q_2$ and $q_3$, $S$ must have a neighbor in $A'$ due to Claim 1. Hence $(Q_1, Q_2)$ is a 2-coloring of $H$.

**Case 2. $D$ is a type 2 dominating set of $I$.**

Suppose $A' = \emptyset$. Then $|B| = 1$ as a result of Claim 1. Let $B = \{S\}$ and $q \in A$. Since $S$ is adjacent to all vertices in $A$, we find that $B' = \emptyset$ as a result of Claim 1. Hence $H$ has no 2-coloring if $|A| = 1$, and $H$ has a 2-coloring $(\{q\}, A \backslash \{q\})$ if $|A| \geq 2$. Suppose $A' \neq \emptyset$. We claim that $(A, A')$ is a 2-coloring of $H$. This can be seen as follows. By definition, each vertex in $\mathcal{S}$ is adjacent to a vertex in $A$. Suppose $|B| = 1$ and let $B = \{S\}$. Since $S$ dominates $Q$ and $A' \neq \emptyset$, $S$ has at least one neighbor in $A'$. Suppose $|B| \geq 2$. Since every vertex in $B$ is adjacent to all vertices in $A$, every vertex in $\mathcal{S}$ must have a neighbor in $A'$ as a result of Claim 1. $\square$

## 5 Conclusions

The key contributions of this paper are the following. We presented a new characterization of the class of $P_6$-free graphs, which strengthens results of Liu and Zhou [14] and Liu, Peng and Zhao [15]. We used an algorithmic technique to prove this characterization. Our main algorithm efficiently finds for any given connected $P_6$-free graph a dominating subgraph that is either an induced $C_6$ or a (not necessarily induced) complete bipartite graph. Besides these main results, we also showed that our characterization is “minimal” in the sense that there exists an infinite family of $P_6$-free graphs for which a smallest connected dominating subgraph is a (not induced) complete bipartite graph. We also characterized the class $Forb(\{C_3^L, P_6\})$ in terms of connected dominating subgraphs, thereby generalizing a result of Bacsó, Michalak and Tuza [2].

Our main algorithm can be useful to determine the computational complexity of decision problems restricted to the class of $P_6$-free graphs. To illustrate this, we applied this algorithm to prove that the Hypergraph 2-Colorability problem is polynomially solvable for the class of hypergraphs with $P_6$-free incidence graphs. Are there any other decision problems for which the algorithm is useful? In recent years, several authors studied the classical $k$-Colorability problem for the class of $P_\ell$-free graphs for various combinations of $k$ and $\ell$ [13, 16, 18]. The 3-Colorability problem is proven to be polynomially solvable for the class of $P_6$-free graphs [16]. Hoàng et al. [13] show that for all fixed $k \geq 3$ the $k$-Colorability problem becomes polynomially solvable for the class of $P_5$-free graphs. They pose the question whether there exists a polynomial time algorithm to determine if a $P_6$-free graph can be 4-colored. We do not know yet if our main algorithm can be used for simplifying the proof of the result in [16] or for solving the open problem described above. We leave these questions for future research.

The next class to consider is the class of $P_7$-free graphs. Recall that a graph $G$ is $P_7$-free if and only if each connected induced subgraph of $G$ contains a dominating subgraph of diameter at most three [1]. Using an approach similar to the one described in this paper, it is possible to find such a dominating subgraph in polynomial time. However, a more important question is whether this characterization of $P_7$-free graphs can be narrowed down. Also determining the computational complexity of the Hypergraph 2-Colorability problem for the class of hypergraphs with $P_7$-free incidence graphs is still an open problem.

Finally, a natural problem for a given graph class deals with its recognition. We are not aware of any recognition algorithms for (even bipartite or triangle-free) $P_7$-free graphs that have a better running time than the trivial algorithm that checks for every 7-tuple of vertices whether they induce a path. This might be another interesting direction for future research, considering the following results on recognition of (subclasses of) $P_6$-free graphs. Fouquet [9] presents a cubic recognition algorithm for the class of $P_6$-free graphs (in an internal report). Giakoumakis and Vanherpe [11] show that bipartite $P_6$-free graphs can be recognized in linear time. They do this by extending the techniques developed in [6] for linear time recognition of $P_4$-free graphs (also see [12]) and by using a characterization of $P_6$-free graphs in terms of canonical decomposition trees (which is not related to our characterization) from [9]. Brandstädt, Klembt and Mahfud [4] show that triangle-free $P_6$-free graphs have bounded clique-width. The recognition algorithm they obtain from this result runs in quadratic time. Since the class of $P_6$-free graphs has unbounded clique-width (cf. [3]), their technique cannot be applied to find a quadratic recognition algorithm for the class of $P_6$-free graphs.

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