Partitioning graphs into connected parts*

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Abstract. The 2-Disjoint Connected Subgraphs problem asks if a given graph has two vertex-disjoint connected subgraphs containing prespecified sets of vertices. We show that this problem is NP-complete even if one of the sets has cardinality 2. The Longest Path Contractibility problem asks for the largest integer ℓ for which an input graph can be contracted to the path P_ℓ on ℓ vertices. We show that the computational complexity of the Longest Path Contractibility problem restricted to P_ℓ-free graphs jumps from being polynomially solvable to being NP-hard at $\ell = 6$, while this jump occurs at $\ell = 5$ for the 2-Disjoint Connected Subgraphs problem. We also present an exact algorithm that solves the 2-Disjoint Connected Subgraphs problem faster than $\mathcal{O}^*(2^n)$ for any n-vertex P_ℓ-free graph. For $\ell = 6$, its running time is $\mathcal{O}^*(1.5790^n)$. We modify this algorithm to solve the Longest Path Contractibility problem for P_6-free graphs in $\mathcal{O}^*(1.5790^n)$ time.

1 Introduction

There are several natural and elementary algorithmic problems that check if the structure of some fixed graph H shows up as a pattern within the structure of some input graph G. One of the most well-known problems is the H-Minor Containment problem that asks whether a given graph G contains H as a minor. A celebrated result by Robertson and Seymour [12] states that the H-Minor Containment problem can be solved in polynomial time for every fixed pattern graph H. They obtain this result by designing an algorithm that solves the following problem in polynomial time for any fixed input parameter k.

Disjoint Connected Subgraphs
Instance: A graph $G = (V, E)$ and mutually disjoint nonempty sets $Z_1, \ldots, Z_t \subseteq V$ such that $\sum_{i=1}^{t} |Z_i| \leq k$.
Question: Do there exist mutually vertex-disjoint connected subgraphs $G_1, \ldots, G_t$ of G such that $Z_i \subseteq V_{G_i}$ for $1 \leq i \leq t$?

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The first problem studied in this paper is the 2-Disjoint Connected Subgraphs problem, which is a restriction of the above problem to $t = 2$.

The *cyclicity* $\eta(G)$ of a connected graph G, introduced by Blum [2], is the largest integer ℓ for which G is contractible to the cycle C_ℓ on ℓ vertices. We introduce a similar concept: the *path contractibility number* $\vartheta(G)$ of a graph G is the largest integer ℓ for which G is P_ℓ-contractible. For convenience, we define $\vartheta(G) = 0$ if and only if G is disconnected. The second problem studied in this paper is the Longest Path Contractibility problem, which asks for the path contractibility number of a given graph G.

Like the 2-Disjoint Connected Subgraphs problem, the Longest Path Contractibility problem deals with partitioning a given graph into connected subgraphs. Since connectivity is a "global" property, both problems are examples of "non-local" problems, which are typically hard to solve exactly (see e.g. [5]). Arguably the most well-known non-local problem is the Travelling Salesman problem, for which no exact algorithm with better time complexity than $\mathcal{O}^*(2^n)$ is known. (The $\mathcal{O}^*$-notation, used throughout the paper, suppresses factors of polynomial order.) Another example of a non-local problem is the Connected Dominating Set problem. The fastest known exact algorithm for the Connected Dominating Set problem runs in $\mathcal{O}^*(1.9407^n)$ time [5], whereas for the general (unconnected) version of the Dominating Set problem an $\mathcal{O}^*(1.5063^n)$ exact algorithm is known [13].

In an attempt to design fast exact algorithms for non-local problems, one can focus on restrictions of the problem to certain graph classes. One family of graph classes of particular interest is the family of graphs that do not contain long induced paths. Several authors have studied restrictions of well-known NP-hard problems, such as the k-Colorability problem (cf. [8, 11, 14]) and the Maximum Independent Set problem (cf. [7, 10]), to the class of P_ℓ-free graphs for several values of ℓ.

Our results. We show that the 2-Disjoint Connected Subgraphs problem is NP-complete even if one of the given sets of vertices has cardinality 2. We also show that the 2-Disjoint Connected Subgraphs problem restricted to the class of P_ℓ-free graphs jumps from being polynomially solvable to being NP-hard at $\ell = 5$, while for the Longest Path Contractibility problem this jump occurs at $\ell = 6$.

A trivial algorithm solves the Two Disjoint Connected Subgraphs problem in $\mathcal{O}^*(2^n)$ time. Let $\mathcal{G}^{k,r}$ denote the class of graphs all connected induced subgraphs of which have a connected r-dominating set of size at most k. We present an algorithm, called SPLIT, that solves the 2-Disjoint Connected Subgraphs problem for n-vertex graphs in the class $\mathcal{G}^{k,r}$ in $\mathcal{O}^*((f(r))^n)$ time for any fixed k and $r \geq 2$, where

$$f(r) = \min_{0<c\leq 0.5}\Big\{\max\Big\{\frac{1}{c^c(1-c)^{1-c}}, 2^{1-\frac{2c}{r-1}}\Big\}\Big\}.$$

In particular, SPLIT solves the 2-Disjoint Connected Subgraphs problem for any n-vertex P_6-free graph in $\mathcal{O}^*(1.5790^n)$ time. We modify SPLIT to obtain an $\mathcal{O}^*(1.5790^n)$ time algorithm for the LONGEST PATH CONTRACTIBILITY problem restricted to P_6-free graphs on n vertices.

2 Preliminaries

All graphs in this paper are undirected, finite, and *simple*, i.e., without loops and multiple edges. We refer to [4] for terminology not defined below.

Let $G = (V, E)$ be a graph. For a subset $S \subseteq V$ we write $G[S]$ to denote the the subgraph of G *induced* by S. We write P_ℓ respectively C_ℓ to denote a path respectively a cycle on ℓ vertices. The *distance* $d_G(u, v)$ between two vertices u and v in a graph G is the *length* $|V_P| - 1$ of a shortest path P between them. For any vertex $v \in V$ and set $S \subseteq V$, we write $d_G(v, S)$ to denote the length of a shortest path from v to S, i.e., $d_G(v, S) := \min_{w \in S}\ d_G(v, w)$. The *neighborhood* of a vertex $u \in V$ is the set $N_G(u) := \{v \in V \mid uw \in E\}$. The set $N_G^r(S) := \{u \in V \mid d_G(u, S) \leq r\}$ is called the *r-neighborhood* of a set S. A set S *r-dominates* a set S' if $S' \backslash S \subseteq N_G^r(S)$. We also say that S r-dominates $G[S']$. A subgraph H of G is an *r-dominating subgraph* of G if V_H r-dominates G. In case $r = 1$, we use "dominating" instead of "1-dominating". A set $S \subseteq V$ is called a *(k, r)-center* of G if $|S| \leq k$ and $N_G^r(S) = V$. A set S is called *connected* if $G[S]$ is connected. The class of graphs all connected induced subgraphs of which have a connected (k, r)-center is denoted by $\mathcal{G}^{k,r}$. The graph G is called a *split graph* if V can be partitioned into a clique and an independent set.

Let $V' \subset V$ and $p, q \in V \backslash V'$. We say that p is *separated* from q by V' if every path in G from p to q contains a vertex of V'. A graph G is called *H-free* for some graph H if G does not contain an induced subgraph isomorphic to H. The *edge contraction* of edge $e = uv$ in G removes the two end-vertices u and v from G, and replaces them by a new vertex that is adjacent to precisely those vertices to which u or v were adjacent. We denote the resulting graph by $G \backslash e$. A graph G is *contractible to* a graph H (graph G is *H-contractible*) if H can be obtained from G by a sequence of edge contractions. An equivalent way of saying that G is H-contractible is that

- for every vertex h in V_H there is a corresponding nonempty subset $W(h) \subseteq V_G$ of vertices in G such that $G[W(h)]$ is connected, and $\mathcal{W} = \{W(h) \mid h \in V_H\}$ is a partition of V_G; we call a set $W(h)$ an *H-witness set* of G for h, and we call $\mathcal{W}$ an *H-witness structure* of G;
- for every $h_i, h_j \in V_H$, there is at least one edge between witness sets $W(h_i)$ and $W(h_j)$ in G if and only if h_i and h_j are adjacent in H.

If for every $h \in V_H$ we contract the vertices in $W(h)$ to a single vertex, then we end up with the graph H. Note that the witness sets $W(h)$ are not uniquely defined in general, since there may be different sequences of edge contractions that lead from G to H. A pair of vertices (u, v) of a graph G is *P_ℓ-suitable* for some integer $\ell \geq 3$ if and only if G has a P_ℓ-witness structure $\mathcal{W}$ with $W(p_1) = \{u\}$ and $W(p_\ell) = \{v\}$, where $P_\ell = p_1 \ldots p_\ell$. See Figure 1 for two different P_4-witness structures and a P_4-suitable pair of a P_4-contractible graph.

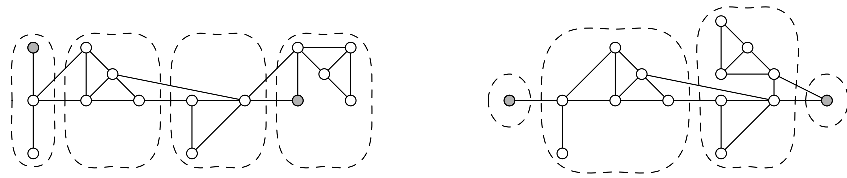

Fig. 1. Two P_4-witness structures of a graph; the grey vertices form a P_4-suitable pair.

A 2-*coloring* of a *hypergraph* $(Q, \mathcal{S})$, where $\mathcal{S}$ is a collection of subsets of Q, is a partition (Q_1, Q_2) of Q with $Q_1 \cap S \neq \emptyset$ and $Q_2 \cap S \neq \emptyset$ for all $S \in \mathcal{S}$.

3 The 2-Disjoint Connected Subgraphs problem

3.1 An NP-completeness proof

Theorem 1. *The 2-Disjoint Connected Subgraphs problem restricted to instances with* $|Z_1| = 2$ *is* NP*-complete.*

Proof. We use a reduction from 3-SAT, which is well-known to be NP-complete (cf. [6]). Let $X = \{x_1, \ldots, x_n\}$ be a set of variables and $C = \{c_1, \ldots, c_m\}$ be a set of clauses forming an instance of 3-SAT. Let $\overline{X} := \{\overline{x} \mid x \in X\}$. We construct a graph G, depicted in Figure 2, as follows. Every literal in $X \cup \overline{X}$ and every clause in C is represented by a vertex in G. There is an edge between $x \in X \cup \overline{X}$ and $c \in C$ if and only if x appears in c. For $i = 1, \ldots, n-1$, x_i and $\overline{x}_i$ are adjacent to both x_{i+1} and $\overline{x}_{i+1}$. We add two vertices f_1 and f_2 to G, where f_1 is adjacent to x_1 and $\overline{x}_1$, and f_2 is adjacent to x_n and $\overline{x}_n$.

We claim that the graph G, together with the sets $Z_1 := \{f_1, f_2\}$ and $Z_2 := C$, is a Yes-instance of the 2-Disjoint Connected Subgraphs problem if and only if C is satisfiable.

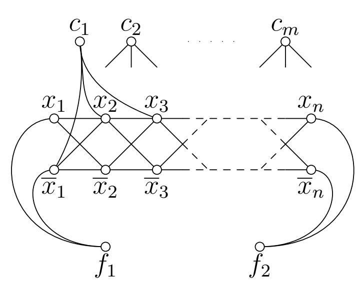

Fig. 2. The graph G, in case $c_1 = (\overline{x}_1 \vee x_2 \vee x_3)$.

Suppose $t : X \to \{\text{true}, \text{false}\}$ is a satisfying truth assignment for C. Let X_T (respectively X_F) be the set of variables that are set to true (respectively false) by t, and let $\overline{X}_T := \{\overline{x} \mid x \in X_T\}$ and $\overline{X}_F := \{\overline{x} \mid x \in X_F\}$. We denote the set of true and false literals by T and F respectively, i.e., $T := X_T \cup \overline{X}_F$ and $F := X_F \cup \overline{X}_T$. Note that exactly one literal of each pair $x_i, \overline{x}_i$ belongs to T, i.e., is set to true by t, and the other one belongs to F. Hence, the vertices in $F \cup \{f_1, f_2\}$ induce a connected subgraph G_1 of G. Since t is a satisfying truth assignment, every clause vertex is adjacent to a vertex in T. Hence the vertices in $T \cup C$ induce a connected subgraph G_2 of G, which is vertex-disjoint from G_1.

To prove the reverse statement, suppose G_1 and G_2 are two vertex-disjoint connected subgraphs of G such that $\{f_1, f_2\} \subseteq V_{G_1}$ and $C \subseteq V_{G_2}$. Since f_1 and f_2 form an independent set in G and G_1 is connected, at least one of each pair $x_i, \overline{x}_i$ must belong to V_{G_1}. Since the vertices of C form an independent set in G, every clause vertex must be adjacent to at least one literal vertex in $(X \cup \overline{X}) \cap V_{G_2}$. Let t be a truth assignment that sets those literals to true, and their negations to false. For each pair $x_i, \overline{x}_i$ both literals of which belong to V_{G_1}, t sets exactly one literal to true, and the other one to false. Then t is a satisfying truth assignment for C. □

3.2 A complexity classification for P_ℓ-free graphs

Consider the following characterization of P_4-free graphs given in [9].

Theorem 2 ([9]). *A graph G is P_4-free if and only if each connected induced subgraph of G contains a dominating induced C_4 or a dominating vertex.*

We use this characterization of P_4-free graphs in the proof of the complexity classification of the 2-Disjoint Connected Subgraphs problem below. Note that we have strengthened the NP-complete cases to split graphs.

Theorem 3. *The* 2-Disjoint Connected Subgraphs *problem is polynomially solvable for P_ℓ-free graphs if $\ell \leq 4$ and* NP*-complete for P_ℓ-free split graphs if $\ell \geq 5$.*

Proof. Assume $\ell \leq 4$. Let $G = (V, E)$ be a P_ℓ-free, and consequently P_4-free, graph with nonempty disjoint sets $Z_1, Z_2 \subseteq V$. Suppose G, together with sets Z_1 and Z_2, is a Yes-instance of the 2-Disjoint Connected Subgraphs problem, and let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be vertex-disjoint connected subgraphs of G such that $Z_i \subseteq V_i$ for $i = 1, 2$. Note that both G_1 and G_2 are P_4-free. As a result of Theorem 2, there exist sets D_1, D_2 such that D_i dominates V_i and $|D_i| \in \{1, 4\}$ for $i = 1, 2$. So to check whether G, together with Z_1 and Z_2, is a Yes-instance of the 2-Disjoint Connected Subgraphs problem, we act as follows.

We guess a vertex $d_1 \in V \backslash Z_2$. If d_1 does not dominate Z_1, we guess another vertex d_1. If d_1 dominates Z_1, we check if Z_2 is contained in one component G_2 of $G[V \backslash (Z_1 \cup \{d_1\})]$. If so, then $G_1 := G[Z_1 \cup \{d_1\}]$ and G_2 form a solution of the 2-Disjoint Connected Subgraphs problem. Otherwise, we choose another vertex d_1. If we have checked every vertex in $V \backslash Z_2$ without finding a solution, then we guess a 4-tuple $D_1 \subseteq V \backslash Z_2$ and repeat the above procedure with D_1 instead of d_1. If we do not find a solution for any 4-tuple D_1, then (G, Z_1, Z_2) is a No-instance of the 2-Disjoint Connected Subgraphs problem. Since we can perform all checks in polynomial time, this finishes the proof of the polynomial cases.

We now show that the 2-Disjoint Connected Subgraphs problem is NP-complete for P_ℓ-free split graphs if $\ell \geq 5$. Clearly, the problem lies in NP. We prove NP-completeness by using a reduction from the NP-complete Hypergraph 2-Colorability problem that asks if a given hypergraph is 2-colorable (cf. [6]). Let $H = (Q, \mathcal{S})$ be a hypergraph with $Q = \{q_1, \ldots, q_n\}$ and $\mathcal{S} = \{S_1, \ldots, S_m\}$. We may assume $m \geq 2$ and $S_i \neq \emptyset$ for each S_i. Let G be the graph obtained from the incidence graph of H by adding the vertices $\mathcal{S}' = \{S'_1, \ldots, S'_m\}$, where $S'_i = S_i$ for every $1 \leq i \leq m$, and by adding the following edges: $q_iS'_j$ if and only if $q_i \in S'_j$, and q_iq_j if and only if $i \neq j$. See Figure 3 for the graph G obtained in this way from the hypergraph $(Q, \mathcal{S})$ with $Q = \{q_1, q_2, q_3\}$ and $\mathcal{S} = \{\{q_1, q_3\}, \{q_1, q_2\}, \{q_1, q_2, q_3\}\}$. Clearly G is a split

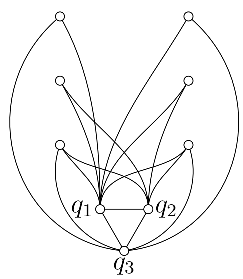

Fig. 3. The graph G.

graph, and it is easy to check that G is P_5-free, and consequently P_ℓ-free for any $\ell \geq 5$. We claim that G, together with the sets $\mathcal{S}$ and $\mathcal{S}'$, is a Yes-instance of the 2-Disjoint Connected Subgraphs problem if and only if $(Q, \mathcal{S})$ has a 2-coloring.

Suppose G_1 and G_2 are vertex-disjoint connected subgraphs of G such that $\mathcal{S} \subseteq V_{G_1}$ and $\mathcal{S}' \subseteq V_{G_2}$. Without loss of generality, assume that $V_1 := V_{G_1}$ and $V_2 := V_{G_2}$ form a partition of V. Then there exists a partition (Q_1, Q_2) of Q such that $V_1 = \mathcal{S} \cup Q_1$ and $V_2 = \mathcal{S}' \cup Q_2$. Note that $\mathcal{S}$ is an independent set in G. Hence $Q_1 \neq \emptyset$ and every vertex in $\mathcal{S}$ is adjacent to at least one vertex in Q_1. Similarly, $Q_2 \neq \emptyset$ and every vertex in $\mathcal{S}'$ has at least one neighbor in Q_2. Since $S'_i = S_i$ for every $1 \leq i \leq m$, (Q_1, Q_2) is a 2-coloring of $(Q, \mathcal{S})$.

Now suppose $(Q, \mathcal{S})$ has a 2-coloring (Q_1, Q_2). Then it is clear that $G[\mathcal{S} \cup Q_1]$ and $G[\mathcal{S}' \cup Q_2]$ are connected, so we can choose $G_1 := G[\mathcal{S} \cup Q_1]$ and $G_2 := G[\mathcal{S}' \cup Q_2]$. This finishes the proof of the NP-complete cases.

3.3 An exact algorithm

Here, we present an algorithm that solves the 2-Disjoint Connected Subgraphs problem for $\mathcal{G}^{k,r}$ for any k and $r \geq 2$ faster than the trivial $\mathcal{O}^*(2^n)$.

Lemma 1. *Let $G = (V, E)$ be a connected induced subgraph of a graph $G' \in \mathcal{G}^{k,r}$. For each subset $Z \subseteq V$, there exists a set $D^* \subseteq V$ with $|D^*| \leq (r-1)|Z|+k$ such that $G[D^* \cup Z]$ is connected.*

Proof. By definition of $\mathcal{G}^{k,r}$, G has a connected (k, r)-center D_0. Let $D_i := \{v \in V \mid d_G(v, D_0) = i\}$ for $i = 1, \ldots r$. Note that the sets $D_0, \ldots, D_r$ form a partition of V. Let z be any vertex of Z and suppose $z \in D_i$ for some $0 \leq i \leq r$; note that this i is uniquely defined. By definition, there exists a path P^z of length i from z to a vertex in D_0, and it is clear that $D_0 \cup P^z \backslash \{z\}$ is a connected set of size $(i-1) + |D_0|$ that dominates z. Let $\mathcal{P} := \bigcup_{z \in Z} P^z \backslash \{z\}$. Clearly, $D^* := D_0 \cup \mathcal{P}$ is a connected set dominating Z. In the worst case, we have $Z \subseteq D_r$ and every pair of paths $P^z, P^{z'}$ is vertex-disjoint, in which case $|D^*| = (r-1)|Z| + |D_0| \leq (r-1)|Z| + k$. This finishes the proof of Lemma 1. □

Lemma 1 implies the following.

Corollary 1. *For any fixed k, the 2-*Disjoint Connected Subgraphs *problem for $\mathcal{G}^{k,r}$ can be solved in polynomial time if $r = 1$, or if one of the given sets Z_1 or Z_2 of vertices has fixed size.*

Proof. Let $G = (V, E)$ be a connected graph in $\mathcal{G}^{k,r}$, and let G together with sets $Z_1, Z_2 \subseteq V$ be an instance of the 2-Disjoint Connected Subgraphs problem. If G, together with the sets Z_1 and Z_2, is a Yes-instance, then G has two vertex-disjoint connected subgraphs G_1, G_2 such that $Z_i \subseteq V_{G_i}$ for $i = 1, 2$. By Lemma 1, there exists a set $D^* \subseteq V_{G_1}$ such that $|D^*| \leq (r-1)|Z_1| + k$ and $G[D^* \cup Z_1]$ is connected. Note that D^* has fixed size k if $r = 1$, and D^* has fixed size $(r-1)|Z_1| + k$ if Z_1 has fixed size. Hence, we can solve the problem in polynomial time by performing the following procedure.

Initially, set $V_1 := Z_1$ and $V_2 := Z_2$. For all sets $Z' \subseteq V \backslash Z_2$ in order of increasing cardinality up to at most $(r-1)|Z_1| + k$, check whether $G[Z' \cup Z_1]$ is connected. If not, choose another set Z'. Otherwise, add Z' to V_1 and check for every vertex $v \in V \backslash (Z' \cup Z_1 \cup Z_2)$ whether v is separated from Z_2 by $Z_1 \cup Z'$. If so, put v in V_1, otherwise put v in V_2. After checking all vertices of $V \backslash (Z' \cup Z_1 \cup Z_2)$, verify whether the graph $G[V_2]$ is connected. If so, the graphs $G_1 := G[V_1]$ and $G_2 := G[V_2]$ form the desired solution. If not, choose another set Z' and repeat the procedure. If no solution is found for any set Z', then no solution to the problem exists.

Since all checks can be done in polynomial time and we only have to perform this procedure a fixed number of times, the 2-Disjoint Connected Subgraphs problem for $\mathcal{G}^{k,r}$ can indeed be solved in polynomial time if $r = 1$, or if one of the given sets of vertices has fixed size. □

From now on, we assume that $r \geq 2$ (and that the sets Z_1, Z_2 may have arbitrary size). We present the algorithm `SPLIT` that solves the 2-Disjoint Connected Subgraphs problem for any $G \in \mathcal{G}^{k,r}$, or concludes that a solution does not exist. We assume $1 \leq |Z_1| \leq |Z_2|$ and define $Z := V \backslash (Z_1 \cup Z_2)$. Algorithm `SPLIT` distinguishes between whether or not Z_1 has a "reasonably" small size, i.e., size at most an for some number $0 < a \leq \frac{1}{2(r-1)}$, the value of which will be determined later.

Case 1. $|Z_1| \leq an$.
For all sets $Z' \subseteq Z$ in order of increasing cardinality up to at most $(r-1)|Z_1|+k$, check whether $G_1 := G[Z' \cup Z_1]$ is connected and $G[(Z \backslash Z') \cup Z_2]$ has a component G_2 containing all vertices of Z_2. If so, output G_1 and G_2. If not, choose another set Z' and repeat the procedure. If no solution is found for any set Z', then output No.

Case 2. $|Z_1| > an$.
Perform the procedure described in Case 1 for all sets $Z' \subseteq Z$ in order of increasing cardinality up to at most $\lceil (1-2a)n \rceil$.

Theorem 4. *For any fixed k and $r \geq 2$, algorithm `SPLIT` solves the 2-Disjoint Connected Subgraphs problem for any n-vertex graph in $\mathcal{G}^{k,r}$ in $\mathcal{O}^*((f(r))^n)$ time, where*

$$f(r) = \min_{0<c\leq 0.5} \Big\{ \max \Big\{ \frac{1}{c^c(1-c)^{1-c}}, 2^{1-\frac{2c}{r-1}} \Big\} \Big\}.$$

Proof. Let $G = (V, E)$ be a graph in $\mathcal{G}^{k,r}$ with $|V| = n$, and let $Z_1, Z_2 \subseteq V$ be two nonempty disjoint sets of vertices of G with $1 \leq |Z_1| \leq |Z_2|$. If Case 1 occurs, the correctness of `SPLIT` follows from Lemma 1. If Case 2 occurs, correctness follows from the fact that all subsets of Z may be checked if necessary, as $|Z_1| > an$ implies $|Z_2| > an$, and therefore $|Z| \leq (1-2a)n$. We are left to prove that the running time mentioned in Theorem 4 is correct. We consider Case 1 and Case 2.

Case 1. $|Z_1| \leq an$.
In the worst case, the algorithm has to check all sets $Z' \subseteq Z$ in order of increasing cardinality up to $(r-1)|Z_1|+k \leq (r-1)an+k$. Let $c := (r-1)a$, and note that $c \leq \frac{1}{2}$ since we assumed $a \leq \frac{1}{2(r-1)}$. Then we must check at most $\sum_{i=1}^{cn+k} \binom{n}{i}$ sets Z'. It is not hard to see that

$$\sum_{i=1}^{cn+k} \binom{n}{i} \leq (cn + (n-cn)^k) \binom{n}{cn}.$$

Using Stirling's approximation, $n! \approx n^n e^{-n} \sqrt{2\pi n}$, we find that the number of sets we have to check is

$$\mathcal{O}\Big(\frac{cn + (n-cn)^k}{\sqrt{2\pi(1-c)cn}} \cdot \Big(\frac{1}{c^c \cdot (1-c)^{1-c}} \Big)^n \Big).$$

For each set all the required checks can be done in polynomial time. Since k is a fixed constant, independent of n, the running time for Case 1 is

$$\mathcal{O}^*\Big(\Big(\frac{1}{c^c \cdot (1-c)^{1-c}}\Big)^n\Big).$$

Case 2. $|Z_1| > an$.
In the worst case, the algorithm has to check all $\mathcal{O}(2^{(1-2a)n})$ sets $Z' \subseteq Z$ in order of increasing cardinality up to $\lceil (1-2a)n \rceil$. Since for each set all the required checks can be done in polynomial time, the running time for Case 2 is

$$\mathcal{O}^*\Big(\Big(2^{1-2a}\Big)^n\Big) = \mathcal{O}^*\Big(\Big(2^{1-\frac{2c}{r-1}}\Big)^n\Big).$$

Since we do not know in advance whether Case 1 or Case 2 will occur, the appropriate value of c can be computed by taking

$$\min_{0<c\leq 0.5}\left\{\max\left\{\frac{1}{c^c \cdot (1-c)^{1-c}}, 2^{1-\frac{2c}{r-1}}\right\}\right\}.$$

This finishes the proof of Theorem 4. □

See Table 1 for the time complexities of `SPLIT` for some graph classes.

Input graph is...	`SPLIT` runs in...
split	$\mathcal{O}^*(1.5790^n)$
P_5-free	$\mathcal{O}^*(1.5790^n)$
P_6-free	$\mathcal{O}^*(1.5790^n)$
P_ℓ-free ($\ell \geq 7$)	$\mathcal{O}^*((f(\ell-3))^n)$
P_7-free	$\mathcal{O}^*(1.7737^n)$
P_8-free	$\mathcal{O}^*(1.8135^n)$
P_{100}-free	$\mathcal{O}^*(1.9873^n)$

Table 1. The time complexities of `SPLIT` for some graph classes.

To prove that the time complexities in Table 1 are correct, we use two results that characterize graphs without long induced paths in terms of connected dominating subgraphs. We presented the following characterization of the class of P_6-free graphs in [9].

Theorem 5 ([9]). *A graph G is P_6-free if and only if each connected induced subgraph of G on more than one vertex contains a dominating induced cycle on six vertices or a dominating (not necessarily induced) complete bipartite subgraph.*

The following result is due to Bacsó and Tuza [1].

Theorem 6 **([1]).** *Let $\ell \geq 7$. A graph G is P_ℓ-free if and only if each connected induced subgraph of G has a dominating subgraph of diameter at most $\ell - 4$.*

Theorem 7. *The time complexities of* `SPLIT` *shown in Table 7 are correct.*

Proof. Since a graph of diameter at most $\ell - 4$ has an $(\ell-4)$-dominating vertex, every P_ℓ-free graph is in $\mathcal{G}^{1,\ell-3}$ for each $\ell \geq 7$ as a result of Theorem 6. Evaluating the function f in Theorem 4 at $r = 4$, $r = 5$ and $r = 97$ yields the running times for P_7-free, P_8-free and P_{100}-free graphs in Table 7. Since $f(2) \approx 1.5790$, it remains to show that both the class of split graphs and the class of P_ℓ-free graphs for $\ell \in \{5, 6\}$ belong to $\mathcal{G}^{k,2}$ for some constant k.

Since every connected induced subgraph of a split graph has a 2-dominating set of size 1 (namely any vertex of the "clique part" of the split graph), the family of split graphs belongs to $\mathcal{G}^{1,2}$. Since every induced C_6 has a dominating connected set of size 4, and every complete bipartite graph has a dominating connected set of size 2, the class of P_6-free graphs is in $\mathcal{G}^{4,2}$ as a result of Theorem 5. The observation that the class of P_5-free graphs is a subclass of the class of P_6-free graphs finishes the proof of Theorem 7. Let G be the graph obtained from a complete graph on vertices $\{x_1, \ldots, x_p\}$ by adding an edge between each x_i and a new vertex y_i, which is only made adjacent to x_i. The graph G is P_5-free, and G does not belong to $\mathcal{G}^{k,1}$ for any constant k. This example shows that we cannot reduce $r = 2$ to $r = 1$ for P_5-free graphs. □

4 The Longest Path Contractibility problem

4.1 A complexity classification for P_ℓ-free graphs

Before stating the main theorem of this section, we first present a number of useful results.

Theorem 8. *The P_4-*Contractibility *problem is* NP*-complete for the class of P_6-free graphs.*

Proof. Brouwer and Veldman [3] give an elegant reduction from the Hypergraph 2-Colorability problem to show that the P_4-Contractibility problem is NP-complete. Given a hypergraph $(Q, \mathcal{S})$ they construct a graph G such that $(Q, \mathcal{S})$ has a 2-coloring if and only if G is P_4-contractible. Hence, to prove Theorem 8, it suffices to show that G is P_6-free. Below we show how G is constructed.

Let $(Q, \mathcal{S})$ be a hypergraph with $Q = \{q_1, \ldots, q_n\}$ and $\mathcal{S} = \{S_1, \ldots, S_m\}$, and assume without loss of generality that $S_m = Q$. The graph $G = (V, E)$ is constructed from the incidence graph of $(Q, \mathcal{S})$ as follows. First we add two new vertices s, s' and a copy $\mathcal{S}' = \{S'_1, \ldots, S'_m\}$ of $\mathcal{S}$, such that $S'_i = S_i$ for every $1 \leq i \leq m$. Then we add the following edges:

- $S_i S'_j$ for every $1 \leq i, j \leq m$;
- sS_i for every $1 \leq i \leq m$;

- $s'S'_i$ for every $1 \leq i \leq m$;
- $S'_i q_j$ if and only if $q_j \in S_i$;
- $q_i q_j$ if and only if $i \neq j$.

See Figure 4 for the graph G obtained in this way from the hypergraph $(Q, \mathcal{S})$ with $Q = \{q_1, q_2, q_3\}$ and $\mathcal{S} = \{\{q_1, q_3\}, \{q_1, q_2\}, \{q_1, q_2, q_3\}\}$. We claim that G

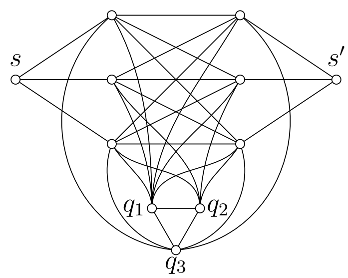

Fig. 4. The graph G.

is P_6-free. This can be seen as follows. Let P be an induced path of G with maximum length over all induced paths of G. Note that P contains at most 2 vertices of Q, since Q is a clique in G. Suppose P starts in s or s'. By symmetry we may assume that P starts in s. Let S_i be the next vertex of P. If P does not contain any vertex of $\mathcal{S}'$, then $V_P \backslash \{s, S_i\} \subseteq Q$ and P has length at most 4. Suppose P contains some vertex S'_j. Then $sS_iS'_j$ is a subpath of P and the next vertex on P is either s' or lies in Q. In the first case $P = sS_iS'_js'$, so P has length 4. In the second case P ends in Q (as $G[\mathcal{S} \cup \mathcal{S}']$ is complete bipartite) and has length at most 5.

Suppose P starts in S_i or S'_i for some $1 \leq i \leq m$ and does not end in s or s'. By symmetry we may assume P starts in S_i. If the second vertex of P is s, then P does not contain any vertex of $\mathcal{S}'$ and has length at most 5. If the second vertex of P is from Q, then P does not contain a vertex from $\mathcal{S}'$. In that case, P either ends in Q and has length at most 3, or P ends in $\mathcal{S}$ and consequently does not contain s or more than two vertices of Q, so P has length at most 4. If the second vertex of P is from $\mathcal{S}'$ and P does not end in this vertex, then P ends in Q and has length at most 4.

Suppose P starts in Q and does not end in a vertex in $\{s, s'\} \cup \mathcal{S} \cup \mathcal{S}'$. Then P ends in Q, and consequently, P has length at most 2. We conclude that G is indeed P_6-free. □

Lemma 2. *For $\ell \geq 3$, a graph G is P_ℓ-contractible if and only if G has a P_ℓ-suitable pair.*

Proof. By definition, G is P_ℓ-contractible if G has a P_ℓ-suitable pair of vertices. To prove the reverse statement, let G be a P_ℓ-contractible graph and let $\mathcal{W}$ be a P_ℓ-witness structure of G. Suppose $|W(p_1)| \geq 2$. Let $x \in W(p_1)$ be a vertex that is not a cutvertex of $G[W(p_1)]$.

Suppose $W(p_1)$ contains a vertex $y \neq x$ adjacent to $W(p_2)$. Then we define $W'(p_1) := \{x\}$, $W'(p_2) := W(p_2) \cup (W(p_1)\backslash\{x\})$ and $W'(p_i) := W(p_i)$ for $i = 3, \ldots, \ell$.

Suppose x is the only vertex of $W(p_1)$ adjacent to $W(p_2)$. As $|W(p_1)| \geq 2$ and $G[W(p_1)]$ is connected, there exists a vertex $y \in W(p_1)\backslash\{x\}$ that is not a cutvertex of $G[W(p_1)]$. We define $W'(p_1) := \{y\}$, $W'(p_2) := W(p_2) \cup (W(p_1)\backslash\{y\})$ and $W'(p_i) := W(p_i)$ for $i = 3, \ldots, \ell$. So given a P_ℓ-witness structure $\mathcal{W}$ of G, we can always find a P_ℓ-witness structure $\mathcal{W}'$ of G with $|W'(p_1)| = 1$. Since $\ell \geq 3$, we did not change the witness sets $W(p_\ell)$ and $W(p_{\ell-1})$ in obtaining $\mathcal{W}'$. Hence, we can repeat the arguments above for $W(p_\ell)$ to obtain a P_ℓ-witness structure $\mathcal{W}''$ of G with $|W''(p_1)| = |W''(p_\ell)| = 1$. By definition, the two vertices of $W''(p_1) \cup W''(p_\ell)$ form a P_ℓ-suitable pair of G. □

Lemma 3. *Let x and y be two neighbors of a vertex u in a graph G with $xy \in E_G$, and let v be some other vertex in G. Then (u, v) is a P_ℓ-suitable pair of G if and only if (u, v) is a P_ℓ-suitable pair of $G\backslash xy$.*

Proof. Suppose (u, v) is a P_ℓ-suitable pair of G. By definition, G has a P_ℓ-witness structure $\mathcal{W}$ with $W(p_1) = \{u\}$ and $W(p_\ell) = \{v\}$. Then $x, y \in N(u)$ are both in the same witness set, namely $W(p_2)$. Hence we may contract edge xy in order to obtain a P_ℓ-witness structure $\mathcal{W}'$ for $G\backslash xy$ with $W'(p_1) = \{u\}$ and $W'(p_\ell) = \{v\}$. The reverse implication is trivial. □

Lemma 4. *For any edge xy of a P_ℓ-free graph G, the graph $G\backslash xy$ is P_ℓ-free.*

Proof. Let $G = (V, E)$ be a P_ℓ-free graph, and let z be the vertex that is being created by contracting the edge $xy \in E$. Suppose $G\backslash xy$ is not P_ℓ-free and let $p_1p_2 \ldots p_\ell$ be an induced P_ℓ in $G\backslash xy$. Since G is P_ℓ-free, we must have $z = p_j$ for some $2 \leq j \leq \ell - 1$. Suppose x is adjacent to both p_{j-1} and p_{j+1} in G. Then the path $p_1 \ldots p_{j-1}xp_{j+1} \ldots p_\ell$ forms an induced P_ℓ in G, a contradiction. Therefore, x, and by symmetry y, cannot be adjacent to both p_{j-1} and p_{j+1} in G. Without loss of generality, assume that $p_{j-1}x \in E$ and $yp_{j+1} \in E$. Then the path $p_1p_2 \ldots p_{j-1}xyp_{j+1} \ldots p_{\ell-1}$ forms an induced P_ℓ in G, contradicting the P_ℓ-freeness of G. □

We now present a polynomial-time algorithm for deciding whether a P_5-free graph is P_4-contractible.

Theorem 9. *The P_4-*Contractibility *problem is solvable in polynomial time for the class of P_5-free graphs.*

Proof. Let $G = (V, E)$ be a connected P_5-free graph. Lemma 2 states that G is P_4-contractible if and only if G contains a P_4-suitable pair (u, v). Since G has $\mathcal{O}(|V|^2)$ pairs (u, v), it suffices to show that we can check in polynomial time whether a given pair (u, v) is P_4-suitable. It follows from the definition of a P_4-witness structure and the P_5-freeness of G that we only need to consider pairs of vertices at distance 3. If there does not exists such a pair, then G is not P_4-contractible. Suppose (u, v) is a pair of vertices of G with $d_G(u, v) = 3$.

Claim 1. We may without loss of generality assume that $N(u)$ and $N(v)$ are independent sets of cardinality at least 2.

We prove Claim 1 as follows. Lemma 3 and Lemma 4 together immediately imply that we may assume $N(u)$ and $N(v)$ to be independent sets. Now suppose that $N(u)$ has cardinality 1, say $N(u) = \{x\}$. It is clear that (u, v) is a P_4-suitable pair of G if and only if $N(v)$ is contained in one component of $G[V\backslash\{u, v, x\}]$, which can be checked in polynomial time. Hence we may assume that $|N(u)| \geq 2$, and by symmetry $|N(v)| \geq 2$.

Claim 2. Let x and x' be two vertices of G such that x is adjacent to a vertex $w \in N(u)$ but not to a vertex $w' \in N(u)$, and x' is adjacent to w' but not to w. Then $N(u) \subseteq N(x) \cup N(x')$.

We prove Claim 2 as follows. Clearly $u \notin \{x, x'\}$. As $N(u)$ is an independent set by Claim 1, u is neither adjacent to x nor to x'. Then $xx' \in E$, since otherwise the path $x'w'uwx$ is an induced P_5 as a result of Claim 1, contradicting the P_5-freeness of G. Now suppose there exists a vertex $w'' \in N(u)$ not in $N(x) \cup N(x')$. Since w' and w'' are not adjacent as a result of Claim 1, the path $w''uw'x'x$ is an induced P_5 in G. This contradiction proves Claim 2.

Claim 3. Suppose G has a P_4-witness structure $\mathcal{W}$ with $W(p_1) = \{u\}$ and $W(p_4) = \{v\}$. Then at least one of the following holds:

1. *there exists a vertex $x \in W(p_2)\backslash N(u)$ with $N(u) \subseteq N(x)$;*
2. *there exist vertices $x, x' \in W(p_2)\backslash N(u)$ with $N(u) \subseteq N(x) \cup N(x')$.*

We prove this claim as follows. Suppose $\mathcal{W}$ is a P_4-witness structure of G with $W(p_1) = \{u\}$ and $W(p_4) = \{v\}$, and suppose condition 1 does not hold. We show that condition 2 must hold. By Claim 1, $N(u)$ is an independent set of G containing at least two vertices. Since $N(u) \subseteq W(p_2)$ and $G[W(p_2)]$ is connected, we know that $W(p_2)\backslash N(u) \neq \emptyset$. Let $x \in W(p_2)\backslash N(u)$ be a vertex such that $|N(u) \cap N(x)|$ is maximal over all vertices in $W(p_2)\backslash N(u)$. Since condition 1 does not hold, there exists a vertex $w' \in N(u)$ that is not adjacent to x. Then w' is adjacent to a vertex $x' \in W(p_2)\backslash(N(u) \cup \{x\})$, as otherwise w' would be an isolated vertex in $G[W(p_2)]$. By choice of x, there exists a vertex $w \in N(u) \cap N(x)$ not adjacent to x'. By Claim 2, $N(u) \subseteq N(x) \cup N(x')$. This finishes the proof of Claim 3.

It remains to prove how we can check in polynomial time whether (u, v) is a P_4-suitable pair of G. If (u, v) is a P_4-suitable pair of G, then by definition G has a P_4-witness structure $\mathcal{W}$ with $W(p_1) = \{u\}$ and $W(p_4) = \{v\}$. Any such witness structure satisfies at least one of the two conditions in Claim 3. We can check in polynomial time if these conditions hold after guessing one vertex (respectively two vertices) in $V\backslash(N(u) \cup N(v) \cup \{u, v\})$. If so, we check in polynomial time if $N(v)$ is contained in one component of the remaining graph (without vertex v). If all our guesses are negative, then (u, v) is not a P_4-suitable pair of G. $\square$

Theorem 8 and Theorem 9 together yield the main result of this section.

Theorem 10. *The* Longest Path Contractibility *problem restricted to the class of P_ℓ-free graphs is polynomially solvable if $\ell \leq 5$ and* NP*-hard if $\ell \geq 6$.*

Proof. First assume $\ell = 5$. Let $G = (V, E)$ be a P_5-free graph. By definition, $\vartheta(G) = 0$ if and only if G is disconnected. Suppose G is connected. Since G does not contain an induced path on more than four vertices, G is clearly not contractible to such a path. Hence we have $\vartheta(G) \leq 4$. By Theorem 9, we can check in polynomial time whether G is P_4-contractible. If so, then $\vartheta(G) = 4$. Otherwise, we check if G has a P_3-suitable pair. This is a necessary and sufficient condition for P_3-contractibility according to Lemma 2. We can perform this check in polynomial time, since two vertices u, v form a P_3-suitable pair of G if and only if u and v are non-adjacent and $G[V \backslash \{u, v\}]$ is connected. If G is P_3-contractible, then $\vartheta(G) = 3$. If G is not P_3-contractible, then we conclude that $\vartheta(G) = 2$ if G has at least two vertices, and $\vartheta(G) = 1$ otherwise.

Now assume $\ell = 6$. Since a graph G is P_4-contractible if and only if $\vartheta(G) \geq 4$ and the P_4-Contractibility problem is NP-complete for P_6-free graphs by Theorem 8, the Longest Path Contractibility problem is NP-hard for P_6-free graphs.

The claim for all other values of ℓ immediately follows from the fact that the class of P_ℓ-free graphs is a subclass of the class of $P_{\ell'}$-free graphs whenever $\ell \leq \ell'$. □

4.2 An exact algorithm

Algorithm `SPLIT` can be extended to an algorithm that solves the Longest Path Contractibility problem for any n-vertex P_6-free graph in $\mathcal{O}^*(1.5790^n)$ time. This extension is described in detail in the proof of the following theorem.

Theorem 11. *The* Longest Path Contractibility *problem for P_6-free graphs on n vertices can be solved in $\mathcal{O}^*(1.5790^n)$ time.*

Proof. Let $G = (V, E)$ be a P_6-free graph with $|V| = n$. By definition, $\vartheta(G) = 0$ if and only if G is disconnected. Suppose G is connected. Since G does not contain an induced path on six vertices, G is clearly not P_6-contractible. Hence $\vartheta(G) \leq 5$. We first show how we can determine in $\mathcal{O}^*(1.5790^n)$ time if $\vartheta(G) = 5$, i.e., if G is P_5-contractible. We do this by modifying the algorithm `SPLIT` such that it decides in $\mathcal{O}^*(1.5790^n)$ time whether a pair (u, v) of vertices of G is P_5-suitable. Note that G has $\mathcal{O}(n^2)$ pairs (u, v) and G is P_5-contractible if and only if G has a P_5-suitable pair (u, v) by Lemma 2. Before we present the modified algorithm, we introduce some additional terminology and prove a useful claim below.

Let u, v be two vertices of G for which we want to decide if they form a P_5-suitable pair. It follows from the definition of a P_5-witness structure and the P_6-freeness of G that we may without loss of generality assume $d_G(u, v) = 4$. We define the *set of midpoints for* (u, v) as $S(u, v) := \{x \in V \mid d_G(x, u) = d_G(x, v) =$

2}. If no confusion is possible, we write $S = S(u,v)$. We define two sets T_1 and T_2 as follows. Set $T_1 = T_1(u,v)$ consists of all vertices in $V\backslash(\{u,v\}\cup N(u)\cup N(v)\cup S)$ that are separated from v by S but are not separated from u by S. Set $T_2 = T_2(u,v)$ consists of all vertices in $V\backslash(\{u,v\}\cup N(u)\cup N(v)\cup S)$ that are separated from u by S but are not separated from v by S. Note that $T_1 \cap T_2 = \emptyset$ and that we can obtain these two sets in polynomial time.

Claim 1. We may without loss of generality assume that $V = \{u,v\} \cup N(u) \cup N(v) \cup S \cup T_1 \cup T_2$.

We prove Claim 1 as follows. Suppose $V' = V\backslash(\{u,v\}\cup N(u)\cup N(v)\cup S\cup T_1\cup T_2)$ is nonempty. By definition of T_1 and T_2, $V' = W_1 \cup W_2$, where W_1 consists of all vertices that are separated from both u and v by S, and W_2 consists of all vertices that are separated from neither u nor v by S.

First suppose $W_1 \neq \emptyset$. Let $x \in W_1$. Note that $S \subseteq W(p_3)$ for any P_5-witness structure $\mathcal{W}$ of G with $W(p_1) = \{u\}$ and $W(p_5) = \{v\}$. Since x is separated from both u and v by S, we must have $x \in W(p_3)$ for any P_5-witness structure $\mathcal{W}$ of G with $W(p_1) = \{u\}$ and $W(p_5) = \{v\}$; otherwise x would be an isolated vertex in $G[W(p_2)]$ or $G[W(p_4)]$, a contradiction. Hence we may contract x with any of its neighbors, which are either in S or which are also separated from both u and v by S. Then (u,v) is P_5-suitable for the resulting (smaller) graph G' if and only if (u,v) is P_5-suitable for G. Furthermore, by Lemma 4, G is P_6-free. Hence we may continue with G'.

Now suppose $W_2 \neq \emptyset$. Let P be a shortest path in G from a vertex in $N(u)$ to a vertex in $N(v)$, containing a vertex in W_2 but not containing any vertex of S (such a path exists, since $W_2 \neq \emptyset$). Then P contains at most one vertex $u' \in N(u)$ and at most one vertex $v' \in N(v)$, as otherwise we can replace P by a shorter path. Consequently, P contains neither u nor v, and we may without loss of generality assume that P starts in u' and ends in v'. Let $x \in W_2 \cap V_P$. If $V_P = \{u', x, v'\}$, then $d_G(x,u) = d_G(x,v) = 2$. This would mean $x \in S$, a contradiction. Hence P contains another vertex $y \notin \{u', x, v'\}$. Then the path $uu'\overrightarrow{P}v'v$ contains at least six vertices. As G is P_6-free, P is not an induced path in G. Hence, $G[V_P]$ contains an edge $st \notin E_P$, where we assume that s occurs before t on the path P from u' to v'. Since $d_G(u,v) = 4$, we have $u'v' \notin E$. This means that at least one of the two vertices s,t is different from u' and v'. We assume without loss of generality that this vertex is s. Then the path $u'\overrightarrow{P}st\overrightarrow{P}v'$ satisfies the requirements but is shorter than P, a contradiction. This proves Claim 1.

We now show how we can modify the algorithm `SPLIT` to determine if (u,v) is a P_5-suitable pair of G. The modified algorithm takes as input the graph $G[V\backslash\{u,v\}]$ with sets $N(u), N(v), S$. It returns YES if G has three connected subgraphs G_1, G_2, G_3 such that $N(u) \subseteq V_{G_1}$, $S \subseteq V_{G_2}$ and $N(v) \subseteq V_{G_3}$, and it returns NO otherwise. The modified algorithm first determines which of the two sets $N(u) \cup N(v)$ and S is the smallest. Since G is P_6-free, Theorem 5 implies that $G \in \mathcal{G}^{4,2}$. Like the original algorithm `SPLIT` for graphs in $\mathcal{G}^{4,2}$, the modified algorithm then distinguishes between whether or not this smallest set has a "reasonably" small size, i.e., size at most an for some number $0 < a \leq \frac{1}{2}$, the value of which will be determined later.

First assume $|N(u)| + |N(v)| \leq |S|$. We distinguish two cases.

Case 1. $|N(u)| + |N(v)| \leq an$.
For all sets $Z' \subseteq T_1 \cup T_2$ in order of increasing cardinality up to at most $|N(u)| + |N(v)| + 4$, check if $G_1 := G[(Z' \cap T_1) \cup N(u)]$ and $G_3 := G[(Z' \cap T_2) \cup N(v)]$ are both connected. If not, choose another set Z'. Otherwise, check whether S is contained in one component G_2 of the graph $G[(S \cup T_1 \cup T_2)\backslash Z']$. If so, conclude that (u, v) is P_5-suitable. If not, choose another set Z' and repeat the procedure. If no solution is found for any set Z', then conclude that (u, v) is not a P_5-suitable pair of G.

Case 2. $|N(u)| + |N(v)| > an$.
Perform the procedure described in Case 1 for all sets $Z' \subseteq T_1 \cup T_2$ in order of increasing cardinality up to at most $\lceil(1 - 2a)n\rceil$.

Now assume $|S| \leq |N(u)| + |N(v)|$. Again, we distinguish two cases.

Case 1. $|S| \leq an$.
For all sets $Z' \subseteq T_1 \cup T_2$ in order of increasing cardinality up to at most $|S| + 4$, check if the graph $G_2 := G[Z' \cup S]$ is connected. If not, choose another set Z'. Otherwise, check whether the graph $G[(N(u) \cup N(v) \cup T_1 \cup T_2)\backslash Z']$ contains two components G_1, G_3 such that $N(u) \subseteq V_{G_1}$ and $N(v) \subseteq V_{G_3}$. If so, conclude that (u, v) is P_5-suitable. If not, choose another set Z' and repeat the procedure. If no solution is found for any set Z', then conclude that (u, v) is not a P_5-suitable pair of G.

Case 2. $|S| > an$.
Perform the procedure described in Case 1 for all sets $Z' \subseteq T_1 \cup T_2$ in order of increasing cardinality up to at most $\lceil(1 - 2a)n\rceil$.

The proof of correctness and the running time analysis are similar to the proof of Theorem 4. Recall that $G \in \mathcal{G}^{4,2}$. Hence we find that $a \approx 0.17054$ is optimal. After checking $\mathcal{O}(n^2)$ pairs of vertices in G on P_5-suitability, we find in $\mathcal{O}^*(1.5790^n)$ time whether G is P_5-contractible or not. If G is P_5-contractible, then $\vartheta(G) = 5$.

Suppose G is not P_5-contractible. We check if G is P_4-contractible. Recall that G is P_4-contractible if and only if G has a P_4-suitable pair (u, v) by Lemma 2. Let $u, v \in V$ be a pair of vertices of G. By Lemma 2 we may assume $d_G(u, v) = 3$. Define $Z_1 := N_G(u)$, $Z_2 := N_G(v)$ and $G' := G[V\backslash\{u, v\}]$. Note that $Z_1 \cap Z_2 = \emptyset$ as $d_G(u, v) = 3$. Furthermore G' is P_6-free as G is P_6-free. Hence (G', Z_1, Z_2) is an instance of the 2-Disjoint Connected Subgraphs problem for P_6-free graphs. By Theorem 7, we can decide in $\mathcal{O}^*(1.5790^n)$ time whether there exist vertex-disjoint subgraphs G_1, G_2 of G such that $Z_i \subseteq V_{G_i}$ for $i = 1, 2$. It is clear that such subgraphs exist if and only if (u, v) is a P_4-suitable pair of G. Since we have to check $\mathcal{O}(n^2)$ pairs (u, v), we can check in $\mathcal{O}^*(1.5790^n)$ time whether or not G is P_4-contractible. If so, then $\vartheta(G) = 4$.

Suppose G is not P_4-contractible. We check if G has a P_3-suitable pair. This is a necessary and sufficient condition for P_3-contractibility according to Lemma 2. We can perform this check in polynomial time, since two vertices u, v form a P_3-suitable pair of G if and only if u, v are non-adjacent and $G[V \backslash \{u, v\}]$ is connected. If G is P_3-contractible, then $\vartheta(G) = 3$. If G is not P_3-contractible, then we conclude that $\vartheta(G) = 2$ if G has at least two vertices, and $\vartheta(G) = 1$ otherwise. $\square$

5 Conclusions

We showed that the 2-Disjoint Connected Subgraphs problem is already NP-complete if one of the given sets of vertices has cardinality 2. We also showed that the 2-Disjoint Connected Subgraphs problem for the class of P_ℓ-free graphs jumps from being polynomially solvable to being NP-hard at $\ell = 5$, while for the Longest Path Contractibility problem this jump occurs at $\ell = 6$.

Our algorithm SPLIT solves the 2-Disjoint Connected Subgraphs problem for P_ℓ-free graphs faster than $\mathcal{O}^*(2^n)$ for any ℓ. We do not know yet how to improve its running time for P_5-free and P_6-free graphs (which are in $\mathcal{G}^{1,2}$ and $\mathcal{G}^{4,2}$, respectively) but expect we can do better for P_ℓ-free graphs with $\ell \geq 7$ (by using a radius argument). The modification of SPLIT solves the Longest Path Contractibility problem for P_6-free graphs in $\mathcal{O}^*(1.5790^n)$ time. Furthermore, SPLIT might be modified into an exact algorithm that solves the Longest Path Contractibility problem for P_ℓ-free graphs with $\ell \geq 7$ as well. The most interesting question however is to find a fast exact algorithm for solving the 2-Disjoint Connected Subgraphs and the Longest Path Contractibility problem for general graphs.

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