The Price of Connectivity for Feedback Vertex Set

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Abstract. Let $\mathrm{fvs}(G)$ and $\mathrm{cfvs}(G)$ denote the cardinalities of a minimum feedback vertex set and a minimum connected feedback vertex set of a graph G, respectively. In general graphs, the ratio $\mathrm{cfvs}(G)/\mathrm{fvs}(G)$ can be arbitrarily large. We study the interdependence between $\mathrm{fvs}(G)$ and $\mathrm{cfvs}(G)$ in graph classes defined by excluding one induced subgraph H. We show that the ratio $\mathrm{cfvs}(G)/\mathrm{fvs}(G)$ is bounded by a constant for every connected H-free graph G if and only if H is a linear forest. We also determine exactly those graphs H for which there exists a constant c_H such that $\mathrm{cfvs}(G) \leq \mathrm{fvs}(G) + c_H$ for every connected H-free graph G, as well as exactly those graphs H for which we can take $c_H = 0$.

1 Introduction

Numerous important graph parameters are defined as the cardinality of a smallest subset of vertices satisfying a certain property. Well-known examples of such parameters include the cardinality of a minimum vertex cover, a minimum dominating set, or a minimum feedback vertex set in a graph. In many cases, requiring the subset of vertices to additionally induce a connected subgraph defines a natural variant of the original parameter. The cardinality of a minimum connected vertex cover or a minimum connected dominating set are just two examples of such parameters that have received considerable interest from both the algorithmic and structural graph theory communities. An interesting question is what effect the additional connectivity constraint has on the value of the graph parameter in question.

One notable graph parameter that has been studied in this context is the vertex cover number $\tau(G)$, defined as the cardinality of a minimum vertex cover of a graph G. The connected variant of this parameter is the connected vertex cover number, denoted by $\tau_c(G)$ and defined as the cardinality of a minimum connected vertex cover in G. The following observation on the interdependence between $\tau(G)$ and $\tau_c(G)$ for connected graphs G is due to Camby et al. [2].

Observation 1 ([2]) *For every connected graph G, it holds that $\tau_c(G) \leq 2 \cdot \tau(G) - 1$.*

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Given a graph class $\mathcal{G}$, the worst-case ratio $\tau_c(G)/\tau(G)$ over all connected graphs G in $\mathcal{G}$ is defined to be the *price of connectivity* for vertex cover for the class $\mathcal{G}$. Observation 1 implies that for general graphs, the price of connectivity for vertex cover is upper bounded by 2, and the class of all paths shows that the bound of 2 is asymptotically sharp [2]. Cardinal and Levy [4], who coined the term "price of connectivity for vertex cover", showed a stronger upper bound of $2/(1+\epsilon)$ for graphs with average degree ϵn. Camby et al. [2] provided forbidden induced subgraph characterizations of graph classes for which the price of connectivity for vertex cover is upper bounded by t, for $t \in \{1, 4/3, 3/2\}$.

The above idea applies to other graph parameters as well. The following observation, due to Duchet and Meyniel [5], shows the interdependence between the connected domination number $\gamma_c(G)$ and the domination number $\gamma(G)$ of a connected graph G.

Observation 2 ([5]) *For every connected graph G, it holds that $\gamma_c(G) \leq 3 \cdot \gamma(G) - 2$.*

Adapting the terminology used above for vertex cover, Observation 2 implies that the price of connectivity for dominating set on general graphs is upper bounded by 3. The class of all paths again shows that this bound is asymptotically sharp. Zverovich [9] proved that for any graph G, it holds that $\gamma_c(H) = \gamma(H)$ for each connected induced subgraph H of G if and only if G is (P_5, C_5)-free, that is, if and only if G does not contain an induced subgraph isomorphic to P_5 or C_5. This implies that the price of connectivity for dominating set is exactly 1 for the class of (P_5, C_5)-free graphs. Camby and Schaudt [3] proved that $\gamma_c(G) \leq \gamma(G) + 1$ for every connected (P_6, C_6)-free graph G, and showed that this bound is best possible. They also obtained a sharp upper bound of 2 on the price of connectivity for dominating set for (P_8, C_8)-free graphs, and showed that the general upper bound of 3 is asymptotically sharp for (P_9, C_9)-free graphs.

A *feedback vertex set* of a graph G is a set F of vertices such that deleting F makes G acyclic, that is, the graph $G - F$ is a forest. The cardinalities of a minimum feedback vertex set and a minimum connected feedback vertex set of a graph G are denoted by $\mathrm{fvs}(G)$ and $\mathrm{cfvs}(G)$, respectively. For any graph class $\mathcal{G}$, we define the price of connectivity for feedback vertex set to be the worst-case ratio $\mathrm{cfvs}(G)/\mathrm{fvs}(G)$ over all connected graphs G in $\mathcal{G}$. In contrast to the aforementioned upper bounds of 2 and 3 on the price of connectivity for vertex cover and dominating set, respectively, the price of connectivity for feedback vertex is not upper bounded by a constant. Graphs consisting of two disjoint cycles that are connected to each other by an arbitrarily long path show that the price of connectivity for feedback vertex set is not even bounded by a constant for planar graphs. Interestingly, Grigoriev and Sitters [6] showed that for planar graphs of minimum degree at least 3, the price of connectivity for feedback vertex set is at most 11. This upper bound of 11 was later improved to 5 by Schweitzer and Schweitzer [8], who also showed that this bound is tight.

Our Results. We study the price of connectivity for feedback vertex set for graph classes characterized by one forbidden induced subgraph H. A graph is called *H-free* if it does not contain an induced subgraph isomorphic to H. We show that the price of connectivity for feedback vertex set is bounded by a constant on the class of H-free graphs if and only if H is a linear forest, that is, a forest of maximum degree at most 2. In fact, we obtain a more refined tetrachotomy result on the interdependence between $\mathrm{fvs}(G)$ and $\mathrm{cfvs}(G)$ for all connected H-free graphs G, depending on the structure of the graph H. In order to formally state our result, we need the following terminology. The *disjoint union* $G + H$ of two vertex-disjoint graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We write sH to denote the disjoint union of s copies of H, and P_n to denote the path on n vertices. A graph class $\mathcal{G}$ is called *identical*, *additive*, or *multiplicative* if for all connected graphs G in $\mathcal{G}$, it holds that $\mathrm{cfvs}(G) = \mathrm{fvs}(G)$, $\mathrm{cfvs}(G) \leq \mathrm{fvs}(G) + c$ for some constant $c \geq 0$, or $\mathrm{cfvs}(G) \leq d \cdot \mathrm{fvs}(G)$ for some constant $d \geq 0$, respectively. Our result can now be formulated as follows.

Theorem 1. *Let H be a graph, and let $\mathcal{G}$ be the class of H-free graphs. Then it holds that*

(i) $\mathcal{G}$ is multiplicative if and only if H is a linear forest;
(ii) $\mathcal{G}$ is additive if and only if H is an induced subgraph of $P_5 + sP_1$ or sP_3 for some $s \geq 0$;
(iii) $\mathcal{G}$ is identical if and only if H is an induced subgraph of P_3.

A graph class $\mathcal{G}$ that is not multiplicative is said to be *unbounded*. If $\mathcal{G}$ is the class of H-free graphs for some graph H, then Theorem 1 implies that $\mathcal{G}$ is unbounded if and only if H contains a cycle or a vertex of degree at least 3.

2 The Proof of Theorem 1

Statements (i), (ii) and (iii) in Theorem 1 follow from Lemmas 1, 3 and 4 below, respectively.

Lemma 1. *Let H be a graph. Then there is a constant d_H such that* $\mathrm{cfvs}(G) \leq d_H \cdot \mathrm{fvs}(G)$ *for every connected H-free graph G if and only if H is a linear forest.*

Proof. First suppose H is a linear forest. Let G be an H-free graph. Note that G is $P_{2|V(H)|}$-free. Let F be a minimum feedback vertex set in G, and let $v \in F$. Let F' be the set obtained from F by adding, for every vertex $u \in F \setminus \{v\}$, all the vertices of a shortest path from u to v in G. Since G is $P_{2|V(H)|}$-free, we find that $|F'| \leq 2|V(H)||F|$. The set F' is a connected feedback vertex set of G, so $\mathrm{cfvs}(G) \leq |F'| \leq 2|V(H)| \cdot \mathrm{fvs}(G)$. This implies that we can take $d_H = 2|V(H)|$.

Before proving the reverse direction, we first introduce a family of graphs that will be used later in the proof. Let C_n denote the cycle on n vertices. For any three integers i, j, k, we define $B_{i,j,k}$ to be the graph obtained from $C_i + C_j$ by choosing a vertex x in C_i and a vertex y in C_j, and adding a path of length k between x and y; if $k = 0$, then we simply identify x and y. The graphs $B_{5,9,4}$ and $B_{3,3,0}$ are depicted in Figure 1. The graph $B_{3,3,0}$ is called the *butterfly*.

Now suppose H is not a linear forest. We distinguish two cases. Suppose H contains a cycle, and let C be a shortest cycle in H; in particular, C is an induced cycle. For any integer ℓ, the graph $B_\ell := B_{|V(C)|+1,|V(C)|+1,\ell}$ is C-free and therefore H-free. The observation that $\mathrm{cfvs}(B_\ell) = \mathrm{fvs}(B_\ell) + \ell - 1$ for every $\ell \geq 1$ shows that no constant d_H exists as described in the lemma. If H does no contain a cycle, then H is a forest. For any integer ℓ, the graph $B_{3,3,\ell}$ is claw-free. Since we assumed that H is not a linear forest, B_ℓ is also H-free. The observation that $\mathrm{cfvs}(B_\ell) = \mathrm{fvs}(B_\ell) + \ell - 1$ for every $\ell \geq 1$ completes the proof of Lemma 1. □

Lemma 2 below exhibits a structural property of sP_3-free graphs that will be used in the proof of Lemma 3 below. The proof of Lemma 2 has been omitted due to page restrictions.

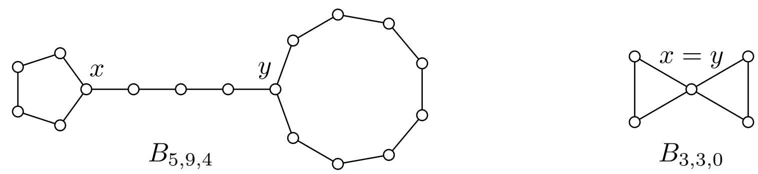

Fig. 1. The graph $B_{5,9,4}$ and the butterfly $B_{3,3,0}$.

Lemma 2. *For every integer s, there is a constant c_s such that* $\mathrm{cfvs}(G) \leq \mathrm{fvs}(G) + c_s$ *for every connected sP_3-free graph G.*

Lemma 3. *Let H be a graph. Then there is a constant c_H such that* $\mathrm{cfvs}(G) \leq \mathrm{fvs}(G) + c_H$ *for every connected H-free graph G if and only if H is an induced subgraph of $P_5 + sP_1$ or sP_3 for some integer s.*

Proof. First suppose H is an induced subgraph of P_5. Let G be a connected H-free graph. In particular, G is P_5-free. Hence, due to a result by Bacsó and Tuza [1], there exists a dominating set $D \subseteq V(G)$ such that D is a clique or D induces a P_3 in G. Let F be a minimum feedback vertex set of G. Note that $|D \setminus F| \leq 2$ if D is a clique and $|D \setminus F| \leq 3$ if D induces a P_3. Since D is a connected dominating set in G, the set $F \cup D$ is a connected feedback vertex set of G of size at most $|F| + 3$. Hence, we can take $c_H = 3$.

Now suppose H is an induced subgraph of $P_5 + sP_1$ for some integer s. Let G be a connected H-free graph. If G is P_5-free, then we can take $c_H = 3$ due to the above arguments. Suppose G contains an induced path P on 5 vertices. Let S be a maximal independent set in the graph obtained from G by deleting the five vertices of P as well as all their neighbors in G. Since G is $P_5 + sP_1$-free, we know that $|S| \leq s - 1$. Note that $V(P) \cup S$ is a dominating set of G. Hence, by Observation 2, there is a connected dominating set D in G of size at most $3(|V(P) \cup S|) - 2 \leq 3s + 10$. Let F be a minimum feedback vertex set in G. Then $F \cup D$ is a connected feedback vertex set in G of size at most $|F| + 3s + 10$. Hence, we can take $c_H = 3s + 10$.

If H is an induced subgraph of sP_3 for some integer s, then the existence of a constant c_H as mentioned in Lemma 3 is guaranteed by Lemma 2.

It remains to show that if H is not an induced subgraph of $P_5 + sP_1$ or sP_3 for any integer s, then there is no constant c_H such that $\mathrm{cfvs}(G) - \mathrm{fvs}(G) + c_H$ for every connected H-free graph G. Let H be a graph that is not an induced subgraph of $P_5 + sP_1$ or sP_3 for any integer s. First suppose H is not a linear forest. Then, by Lemma 1, there does not exist a constant c such that $\mathrm{cfvs}(G) \leq c \cdot \mathrm{fvs}(G)$ for every connected H-free graph G. This implies that there cannot be a constant c_H such that $\mathrm{cfvs}(G) \leq \mathrm{fvs}(G) + c_H$ for every connected H-free graph G. Finally, suppose H is a linear forest. Since H is not an induced subgraph of $P_5 + sP_1$ or sP_3 for any integer s, it contains P_6 or $P_4 + P_2$ as an induced subgraph. Consequently, the class of H-free graphs is a superclass of the class of $\{P_6, P_4 + P_2\}$-free graphs. Hence, in order to complete the proof of Lemma 3, it suffices to show that if $\mathcal{G}$ is the class of $\{P_6, P_4 + P_2\}$-free graphs, then there exists no constant c_H such that $\mathrm{cfvs}(G) \leq \mathrm{fvs}(G) + c_H$ for every connected $G \in \mathcal{G}$.

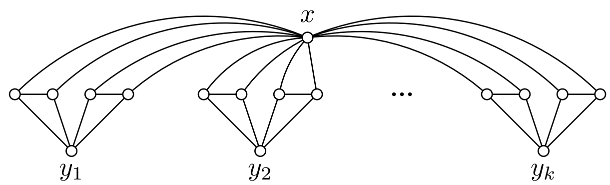

Fig. 2. The graph L_k, defined for every $k \geq 1$.

For every integer $k \geq 1$, let L_k be the graph obtained from k disjoint copies of the butterfly by adding a new vertex x that is made adjacent to all vertices of degree 2; see Figure 2 for an illustration. For every $k \geq 1$, the unique minimum feedback vertex set in L_k is the set $\{x, y_1, y_2, \ldots, y_k\}$, so $\mathrm{fvs}(L_k) = k+1$. Every minimum connected feedback vertex set in L_k contains the set $\{x, y_1, y_2, \ldots, y_k\}$, as well as exactly one additional vertex for each of the vertices y_i to make this set connected. Hence, $\mathrm{cfvs}(L_k) = 2k+1 = \mathrm{fvs}(L_k) + k$. The observation that L_k is $\{P_6, P_4 + P_2\}$-free for every $k \geq 1$ implies that if $\mathcal{G}$ is the class of $\{P_6, P_4 + P_2\}$-free graphs, then there exists no constant c such that $\mathrm{cfvs}(G) \leq \mathrm{fvs}(G) + c$ for every connected $G \in \mathcal{G}$. $\square$

The proof of the next lemma has been omitted due to page restrictions.

Lemma 4. *Let H be a graph. Then $\mathrm{cfvs}(G) = \mathrm{fvs}(G)$ for every connected H-free graph G if and only if H is an induced subgraph of P_3.*

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