Ramsey numbers for graph classes*

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Abstract. For any graph class $\mathcal{G}$ and any two positive integers i and j, the Ramsey number $R_{\mathcal{G}}(i,j)$ is the smallest positive integer such that every graph in $\mathcal{G}$ on at least $R_{\mathcal{G}}(i,j)$ vertices has a clique of size i or an independent set of size j. For the class of all graphs, Ramsey numbers are notoriously hard to determine, and the exact values are known only for very small values of i and j. For planar graphs, a formula for determining all Ramsey numbers was obtained by Steinberg and Tovey (J. Comb. Theory Ser. B 59 (1993), 288–296). On the other hand, it is highly unlikely that such a formula will ever be found for claw-free graphs, as there exist infinitely many nontrivial Ramsey numbers for claw-free graphs that are as difficult to determine as for arbitrary graphs. Inspired by this contrast between known results on planar graphs and claw-free graphs, we initiate a systematic study of Ramsey numbers for graph classes. We give exact formulas for determining all Ramsey numbers for the class of perfect graphs and many of its subclasses. We also establish all Ramsey numbers for line graphs and long circular interval graphs, identified by Chudnovsky and Seymour as "the two principal basic classes of claw-free graphs", as well as for fuzzy circular interval graphs, another important subclass of claw-free graphs that contains all long circular interval graphs. As complementary results, we determine all Ramsey numbers for circular-arc graphs and cactus graphs.

1 Introduction

Ramsey Theory is an important subfield of combinatorics that studies how large a system must be in order to ensure that it contains some particular structure. Since the start of the field in 1930 [21], there has been a tremendous interest in Ramsey Theory, leading to many results as well as several surveys and books (see, e.g., [17] and [20]). For every pair of positive integers i and j, the *Ramsey number* $R(i,j)$ is the smallest positive integer such that every graph on at least $R(i,j)$ vertices contains a clique of size i or an independent set of size j. Ramsey's Theorem [21], in its graph-theoretic version, states that the number $R(i,j)$ exists for every pair of positive integers i and j. As discussed by Diestel ([11], p. 252), this result might seem surprising at first glance. Even more surprising is how difficult it is to determine these values exactly. Despite the vast amount of results that have been produced on Ramsey Theory during the past 80 years, the exact value of $R(i,j)$ is known only for very few small values of i and j. In fact, no more than 16 nontrivial Ramsey numbers have been determined exactly (see Table 1).

Confronted with such difficulty, it is natural to restrict the set of considered graphs. For any graph class $\mathcal{G}$ and any pair of positive integers i and j, we define $R_{\mathcal{G}}(i,j)$ to be the smallest positive integer such that every graph in $\mathcal{G}$ on at least $R_{\mathcal{G}}(i,j)$ vertices contains a clique of size i or an independent set of size j. To the best of our knowledge, Ramsey numbers of this type have been studied previously only for planar graphs, graphs with small maximum degree, and claw-free graphs. Let us briefly summarize the known results on these classes.

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i \ j	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40–42
4	9	18	25	36–41	49–61	56–84	73–115	92–149
5	14	25	43–49	58–87	80–143	101–216	126–316	144–442
6	18	36–41	58–87	102–165	113–298	132–495	169–780	179–1171
7	23	49–61	80–143	113–298	205–540	217–1031	241–1713	289–2826
8	28	56–84	101–216	132–495	217–1031	282–1870	317–3583	330–6090
9	36	73–115	126–316	169–780	241–1713	317–3583	565–6588	581–12677
10	40–42	92–149	144–442	179–1171	289–2826	330–6090	581–12677	798–23556

Table 1. Trivially, it holds that $R(1, j) = 1$ and $R(2, j) = j$ for all $j \geq 1$, and $R(i, j) = R(j, i)$ for all $i, j \geq 1$. The above table contains the currently best known upper and lower bounds on $R(i, j)$ for all $i, j \in \{3, \ldots, 10\}$ [5, 12, 15, 20]. In particular, it contains all nontrivial Ramsey numbers whose exact values are known.

Planar graphs form the only graph class for which *all* Ramsey numbers have been determined exactly. Let $\mathcal{P}$ denote the class of planar graphs. Trivially, $R_{\mathcal{P}}(1, j) = R_{\mathcal{P}}(i, 1) = 1$ for all $i, j \geq 1$. The following theorem establishes all other Ramsey numbers for planar graphs. The theorem is due to Steinberg and Tovey [23], and its proof relies on the famous four color theorem.

Theorem 1 ([23]). *Let $\mathcal{P}$ be the class of planar graphs. Then*

- $R_{\mathcal{P}}(2, j) = j$ *for all* $j \geq 2$,
- $R_{\mathcal{P}}(3, j) = 3j - 3$ *for all* $j \geq 2$,
- $R_{\mathcal{P}}(i, j) = 4j - 3$ *for all* $i \geq 4$ *and* $j \geq 2$ *such that* $(i, j) \neq (4, 2)$, *and*
- $R_{\mathcal{P}}(4, 2) = 4$.

It is interesting to note that almost 25 years before Steinberg and Tovey published their result, Walker [24] established exact values and bounds on all Ramsey numbers for planar graphs, using Heawood's five color theorem. In fact, Walker proved the exact values of all Ramsey numbers for planar graphs, assuming the validity of the –then– four color conjecture.

For any positive integer k, let $\mathcal{G}_k$ be the class of graphs with maximum degree at most k. Staton [22] calculated the exact value of $R_{\mathcal{G}_3}(3, j)$ for all $j \geq 1$, while the Ramsey numbers $R_{\mathcal{G}_3}(4, j)$ for all $j \geq 1$ were obtained by Fraughnaugh and Locke [14]. In the same paper, Fraughnaugh and Locke [14] also determined the exact value of $R_{\mathcal{G}_4}(4, j)$ for all $j \geq 1$, while the numbers $R_{\mathcal{G}_4}(3, j)$ for all $j \geq 1$ had previously been obtained by Fraughnaugh Jones [13].

The only other graph class that has been studied in this context is the class $\mathcal{C}$ of claw-free graphs. Matthews [18] proved exact values as well as upper and lower bounds on some Ramsey numbers for claw-free graphs. In particular, he established the exact value of $R_{\mathcal{C}}(3, j)$ for all $j \geq 1$. Perhaps more interestingly, he proved that $R_{\mathcal{C}}(i, 3) = R(i, 3)$ for all $i \geq 1$. Since the exact value of $R(i, 3)$ is unknown for every $i \geq 10$ (see also Table 1), this implies that there is little hope of finding a formula for determining all Ramsey numbers for claw-free graphs.

Inspired by the contrast between the aforementioned positive result on planar graphs and the negative result on claw-free graphs, we initiate a systematic study of Ramsey numbers for graph classes. Our goal is to identify graph classes for which all Ramsey numbers can be determined exactly. In Section 3, we give a formula for determining all Ramsey numbers for perfect graphs, and show that the same formula yields all Ramsey numbers for every subclass of perfect graphs that contains all disjoint unions of complete graphs, such as chordal graphs, interval graphs, proper interval graphs, permutation graphs, comparability graphs, co-comparability graphs, and cographs. We also determine all Ramsey numbers for other well-known subclasses of perfect graphs, namely split graphs, threshold graphs, bipartite graphs and forests.

Our positive results on subclasses of perfect graphs naturally raise the question for which classes $\mathcal{G}$ it is possible to determine all Ramsey numbers $R_{\mathcal{G}}(i,j)$ in case $\mathcal{G}$ is *not* a subclass of perfect graphs. As mentioned above, the class of planar graphs is an example of such a graph class [23], whereas the class of claw-free graphs is most likely not [18]. In Section 4, we show that this negative result for claw-free graphs also holds for several other graph classes, such as triangle-free graphs, AT-free graphs, and P_5-free graphs, to name but a few. Note that despite these negative results, we are able to determine all Ramsey numbers for bipartite graphs, an important subclass of triangle-free graphs, and for co-comparability graphs, a large subclass of AT-free graphs. Similarly, the negative result on P_5-free graphs contrasts the positive results on split graphs and cographs, two famous subclasses of P_5-free graphs. In other words, our results narrow the gap between known graph classes for which all Ramsey numbers can be determined by exact formulas, and known graph classes for which it is highly unlikely that such a formula will ever be found.

In search for more positive results, we study subclasses of claw-free graphs in Section 4. Recently, Chudnovsky and Seymour [8] proved that every claw-free graph can be composed from graphs belonging to some basic classes. In [9], they identified line graphs and so-called long circular interval graphs as the two principal basic classes of claw-free graphs. In Section 4, we determine all Ramsey numbers for line graphs and long circular interval graphs, as well as for fuzzy circular interval graphs, a superclass of long circular interval graphs. Finally, in Section 5, we give formulas for determining all Ramsey numbers for cactus graphs and circular-arc graphs.

2 Preliminaries

All graphs we consider are undirected, finite and simple. A subset S of vertices of a graph is a *clique* if all the vertices in S are pairwise adjacent, and S is an *independent set* if no two vertices of S are adjacent. For any graph class $\mathcal{G}$ and any two positive integers i and j, we define the Ramsey number $R_{\mathcal{G}}(i,j)$ to be the smallest positive integer such that every graph in $\mathcal{G}$ on at least $R_{\mathcal{G}}(i,j)$ vertices contains a clique of size i or an independent set of size j. When $\mathcal{G}$ is the class of all graphs, we write $R(i,j)$ instead of $R_{\mathcal{G}}(i,j)$. It is well-known that Ramsey numbers for general graphs are symmetric, i.e., that $R(i,j) = R(j,i)$ for all $i,j \geq 1$. More generally, $R_{\mathcal{G}}(i,j) = R_{\mathcal{G}}(j,i)$ for all $i,j \geq 1$ for every class $\mathcal{G}$ that is closed under taking complements, i.e., if for every graph G in $\mathcal{G}$, its complement $\overline{G}$ also belongs to $\mathcal{G}$. If $\mathcal{G}$ is not closed under taking complements, then the Ramsey numbers for $\mathcal{G}$ are typically not symmetric. For any two graph classes $\mathcal{G}$ and $\mathcal{G}'$ such that $\mathcal{G} \subseteq \mathcal{G}'$, we clearly have that $R_{\mathcal{G}}(i,j) \leq R_{\mathcal{G}'}(i,j)$ for all $i,j \geq 1$. In particular, it holds that $R_{\mathcal{G}}(i,j) \leq R(i,j)$ for any graph class $\mathcal{G}$ and all $i,j \geq 1$, which implies that all such numbers $R_{\mathcal{G}}(i,j)$ exist.

The following observation holds for all the graph classes studied in this paper, and for the class of all graphs in particular.

Observation 1. *For any graph class $\mathcal{G}$, $R_{\mathcal{G}}(1,j) = 1$ for all $j \geq 1$. Moreover, if $\mathcal{G}$ contains all edgeless graphs, then $R_{\mathcal{G}}(2,j) = j$ for all $j \geq 1$.*

For some graph classes, we will also make use of the following observation.

Observation 2. *For any graph class $\mathcal{G}$, $R_{\mathcal{G}}(i,1) = 1$ for all $i \geq 1$. Moreover, if $\mathcal{G}$ contains all complete graphs, then $R_{\mathcal{G}}(i,2) = i$ for all $i \geq 1$.*

We refer to the monograph by Diestel [11] for any standard graph terminology not defined below. Let $G = (V, E)$ be a graph, let $v \in V$ and $S \subseteq V$. The complement of G is denoted by $\overline{G}$. We write $G[S]$ to denote the subgraph of G induced by S. For notational convenience, we sometimes write $G - S$ instead of $G[V \setminus S]$ and $G - v$ instead of $G[V \setminus \{v\}]$. The maximum degree of G is denoted by $\Delta(G)$. The clique number $\omega(G)$ of G is the size of a largest clique in G, and the independence number $\alpha(G)$ of G is the size of a largest independent set in G. We write $\nu(G)$ to denote the size of a maximum matching in G, and $\chi(G)$ to denote the chromatic number of G. Given graphs $G_1 = (V_1, E_1), \ldots, G_k = (V_k, E_k)$ such that $V_i \cap V_j = \emptyset$ and $E_i \cap E_j = \emptyset$ for every $i, j \in \{1, \ldots, k\}$ with $i \neq j$, the *disjoint union* of $G_1, \ldots, G_k$ is the graph $(V_1 \cup \ldots \cup V_k, E_1 \cup \ldots \cup E_k)$. The complete graph on ℓ vertices is denoted by K_ℓ. We use P_ℓ and C_ℓ to denote the graphs that are isomorphic to the path and the cycle on ℓ vertices, respectively, i.e., P_ℓ is the graph with vertex set $\{v_1, v_2, v_3, \ldots, v_\ell\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{\ell-1}v_\ell\}$, and C_ℓ is obtained from P_ℓ by adding the edge $v_\ell v_1$.

We now give a brief definition of most of the graph classes studied in this paper. Some graph classes whose definitions require additional terminology will be defined in later sections. For many of the classes mentioned here, several equivalent definitions and characterizations are known; we only mention those that best fit our purposes. Figure 1 shows the inclusion relationship between the classes mentioned in this paper and summarizes our results. More information on these classes, including a wealth of information on applications, can be found in the excellent monographs by Brandstädt et al. [4] and by Golumbic [16].

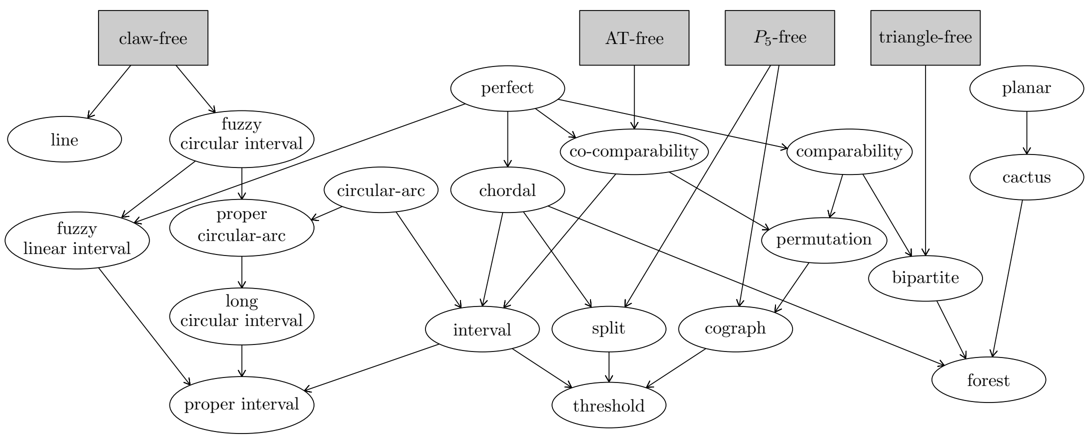

Fig. 1. An overview of the graph classes mentioned in this paper. An arrow from a class $\mathcal{G}_1$ to a class $\mathcal{G}_2$ indicates that $\mathcal{G}_2$ is a proper subclass of $\mathcal{G}_1$. All the depicted inclusion relations were previously known, apart from one: we prove in Lemma 3 that every fuzzy linear interval graph is perfect. For each of the graph classes in an elliptic frame, there exists a formula for determining all Ramsey numbers. Prior to our work, such a formula was only known for planar graphs [23]. For each of the graph classes in a shaded rectangular box, such a formula is unlikely to be found, as there are infinitely many nontrivial Ramsey numbers that are as hard to determine as for general graphs. This was previously known only for claw-free graphs [18].

Let us first define the subclasses of perfect graphs that appear in Section 3. A graph is *perfect* if $\omega(G') = \chi(G')$ for every induced subgraph G' of G. The strong perfect graph theorem, proved by Chudnovsky et al. [7] after being conjectured by Berge more than 40 years earlier, states that a graph is perfect if and only if it does not contain a chordless cycle of odd length at least 5 or the complement of such a cycle as an induced subgraph. A graph is *chordal* if it does not contain a chordless cycle of length greater than 3 as an induced subgraph. A graph $G = (V, E)$ is an *interval graph* if it is the intersection graph of a family $\mathcal{I}$ of intervals of the real line, i.e., if there exists a family $\mathcal{I}$ of intervals of the real line such that one can associate with each vertex $v \in V$ an interval in $\mathcal{I}$ and such that two vertices of G are adjacent if and only if their corresponding intervals intersect; the pair $(G, \mathcal{I})$ is called an *interval model* of G. If a graph G admits an interval model $(G, \mathcal{I})$ such that no interval of $\mathcal{I}$ properly contains another, then G is a *proper interval graph*. A *comparability* graph is a graph that is transitively orientable, i.e., its edges can be directed such that whenever (a, b) and (b, c) are directed edges, then (a, c) is a directed edge. A graph is a *co-comparability* graph if it is the complement of a comparability graph. A *permutation graph* is the intersection graph of a family of line segments connecting two parallel lines; the class of permutation graphs is exactly the intersection between the classes of comparability and co-comparability graphs. A graph is a *cograph* if and only if it does not contain an induced subgraph isomorphic to P_4. A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set. A graph G is a *threshold* graph if and only if there is an ordering $v_1, \ldots, v_n$ of its vertices such that $N_G[v_1] \subseteq N_G[v_2] \subseteq \cdots \subseteq N_G[v_n]$; it is well-known that every threshold graph is a split graph. A graph is *bipartite* if its vertex set can be partitioned into two independent sets.

We now define graph some graph classes that are not subclasses of perfect graphs. For every fixed graph H, the class of *H-free* graphs is the class of graphs that do not contain an induced subgraph isomorphic to H. The *claw* is the graph isomorphic to $K_{1,3}$ and the *triangle* is the graph isomorphic to K_3. An *asteroidal triple (AT)* is a set of three pairwise non-adjacent vertices such that between every two of them, there is a path that does not contain a neighbor of the third. A graph is *AT-free* if it does not contain an AT. A graph $G = (V, E)$ is a *circular-arc graph* if there exists a family $\mathcal{I}$ of arcs of a circle $\mathcal{C}$ such that one can associate with each vertex $v \in V$ an arc in $\mathcal{I}$ and such that two vertices of G are adjacent if and only if their corresponding arcs intersect. The pair $(G, \mathcal{I})$ is called a *circular-arc model* of G. A *proper circular-arc graph* is a circular-arc graph G that has an circular-arc model $(G, \mathcal{I})$ in which no arc of $\mathcal{I}$ properly contains another.

3 Ramsey numbers for subclasses of perfect graphs

In this section, we give exact values for all Ramsey numbers for the class of perfect graphs and several of its subclasses. As we will see below, the strong relationship between the size of a maximum clique and the chromatic number of a perfect graph will be very helpful in determining all Ramsey numbers for subclasses of perfect graphs.

A graph class $\mathcal{G}$ is *χ-bounded* if there exists a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathcal{G}$, we have $\chi(G') \leq f(\omega(G'))$ for every induced subgraph G' of G. Such a function f is called a *χ-bounding function* for $\mathcal{G}$, and we say that $\mathcal{G}$ is *χ-bounded* if there exists a χ-bounding function for $\mathcal{G}$. Both Walker [24] and Steinberg and Tovey [23] observed the close relationship between the chromatic number and Ramsey number of a graph when they studied Ramsey numbers for planar graphs. Their key observation can be applied to any χ-bounded graph class as follows.

Lemma 1. *Let $\mathcal{G}$ be a χ-bounded graph class with χ-bounding function f. Then $R_{\mathcal{G}}(i, j) \leq f(i-1)(j-1)+1$ for all $i, j \geq 1$.*

Proof. Let G be a graph in $\mathcal{G}$ with at least $f(i-1)(j-1)+1$ vertices. Suppose that G contains no K_i. Since G has no K_i, we have $\omega(G) \leq i-1$. By the definition of a χ-bounding function, $\chi(G) \leq f(\omega(G)) \leq f(i-1)$. Let ϕ be any proper vertex coloring of G. Since ϕ uses at most

$f(i-1)$ colors and G has at least $f(i-1)(j-1)+1$ vertices, there must be a color class that contains at least j vertices. Consequently, G contains an independent set of size j. ◻

Theorem 2. *Let $\mathcal{G}$ be a subclass of perfect graphs that contains all disjoint unions of complete graphs. Then $R_{\mathcal{G}}(i,j) = (i-1)(j-1)+1$ for all $i,j \geq 1$. In particular, this holds when $\mathcal{G}$ is the class of perfect graphs, chordal graphs, interval graphs, proper interval graphs, comparability graphs, co-comparability graphs, permutation graphs, or cographs.*

Proof. Observe that the identity function is a χ-bounding function for the class of perfect graphs, and hence also for $\mathcal{G}$. Consequently, $R_{\mathcal{G}}(i,j) \leq (i-1)(j-1)+1$ for all $i,j \geq 1$ due to Lemma 1. The matching lower bound follows from the observation that the disjoint union of $j-1$ copies of K_{i-1} is a graph on $(i-1)(j-1)$ vertices that belongs to $\mathcal{G}$ and that has neither a clique of size i nor an independent set of size j. ◻

We now give formulas for determining all Ramsey numbers for some perfect graph classes that do not contain all disjoint unions of complete graphs. We start with split graphs and threshold graphs, two well-known subclasses of chordal graphs. Recall that, for $i \in \{1,2\}$ and every $j \geq 1$, the Ramsey numbers $R_{\mathcal{G}}(i,j)$ for any graph class $\mathcal{G}$ considered in this paper immediately follow from Observation 1. Note that Observation 2 yields all Ramsey numbers $R_{\mathcal{G}}(i,j)$ with $j \in \{1,2\}$ and $i \geq 1$ when $\mathcal{G}$ is the class of split graphs or threshold graphs, since both classes contain all complete graphs. The following two theorems establish all other Ramsey numbers for these two graphs classes.

Theorem 3. *Let $\mathcal{G}$ be the class of split graphs. Then $R_{\mathcal{G}}(i,j) = i+j-1$ for all $i,j \geq 3$.*

Proof. Let G be a split graph on at least $i+j-1$ vertices whose vertices are partitioned into a clique C and an independent set I. Since $|V(G)| \geq i+j-1$, it is not possible that $|C| < i$ and $|I| < j$, which implies that G contains a clique of size i or an independent set of size j. Hence $R_{\mathcal{G}}(i,j) \leq i+j-1$. For the lower bound, consider a split graph G whose vertices are partitioned into a clique C of size $i-1$ and an independent set I of size $j-1$, such that C is a maximal clique in G, and every vertex $v \in C$ has at least one neighbor in I. Note that such a graph G exists due to the assumption that $i,j \geq 3$. This graph G has $i+j-2$ vertices, and G contains neither a clique of size i nor an independent set of size j. ◻

Theorem 4. *Let $\mathcal{G}$ be the class of threshold graphs. Then $R_{\mathcal{G}}(i,j) = i+j-2$ for all $i,j \geq 3$.*

Proof. Let G be a threshold graph on at least $i+j-2$ vertices whose vertices are partitioned into a clique C and an independent set I, such that I is a maximal independent set. Then there is a vertex $v \in I$ that is adjacent to all the vertices in C. We claim that G contains a clique of size i or an independent set of size j. This is clearly the case if $|C| \geq i-1$, since $C \cup \{v\}$ is a clique. Suppose $|C| \leq i-2$. Then, since $|C| + |I| = |V(G)| \geq i+j-2$, we know that $|I| \geq j$. This implies that $R_{\mathcal{G}}(i,j) \leq i+j-2$ for all $i,j \geq 3$.

To prove the matching lower bound, let G be the threshold graph obtained from a clique of size $i-2$ by adding $j-1$ independent vertices, and making every vertex of the clique adjacent to each of these independent vertices. It is easy to verify that G contains neither a clique of size i nor an independent set of size j. Since $|V(G)| = i+j-3$, we conclude that $R_{\mathcal{G}}(i,j) \geq i+j-2$ and hence $R_{\mathcal{G}}(i,j) = i+j-2$ for all $i,j \geq 3$. ◻

We conclude this section by considering two famous subclasses of perfect graphs to which Obervation 2 does not apply.

Theorem 5. *Let $\mathcal{G}$ be the class of bipartite graphs or the class of forests. Then $R_{\mathcal{G}}(i,j) = 2j-1$ for all $i \geq 3$ and $j \geq 1$.*

Proof. We first prove the theorem when $\mathcal{G}$ is the class of bipartite graphs. Suppose that G is a bipartite graph on at least $2j-1$ vertices with partition (A, B) of its vertices into two independent sets. Then A or B contains at least j vertices. Hence $R_{\mathcal{G}}(i, j) \leq 2j-1$ for all $i \geq 3$ and $j \geq 1$ when $\mathcal{G}$ is the class of bipartite graphs. Since forests are bipartite, this upper bound applies also when $\mathcal{G}$ is the class of forests. For the lower bound, consider a path on $2j-2$ vertices, which is a forest and hence a bipartite graph on $2j-2$ vertices containing neither a clique of size i nor an independent set of size j. □

We now turn our attention to graph classes that are not necessarily perfect. In particular, in the next section, we establish all Ramsey numbers for some important subclasses of claw-free graphs.

4 Ramsey numbers for subclasses of claw-free graphs

Matthews [18] showed that when $\mathcal{G}$ is the class of claw-free graphs, $R_{\mathcal{G}}(i, 3) = R(i, 3)$ for every positive integer i, which implies that there are infinitely many (nontrivial) Ramsey numbers for claw-free graphs that are as hard to determine as for arbitrary graphs. The next theorem implies that this is the case for many other graph classes as well.

Theorem 6. *Let $\mathcal{G}$ be a class of graphs. If $\mathcal{G}$ contains the class of K_i-free graphs as a subclass for some i, then $R_{\mathcal{G}}(i, j) = R(i, j)$ for all $j \geq 1$. Moreover, if $\mathcal{G}$ contains the class of $\overline{K_j}$-free graphs as a subclass for some j, then $R_{\mathcal{G}}(i, j) = R(i, j)$ for all $i \geq 1$.*

Proof. Let i be an integer, and suppose that $\mathcal{G}$ contains the class of K_i-free graphs as a subclass. Clearly, $R_{\mathcal{G}}(i, j) \leq R(i, j)$ for all $j \geq 1$. We now show that $R_{\mathcal{G}}(i, j) \geq R(i, j)$ for all $j \geq 1$. For every integer $j \geq 1$, there exists, by the definition $R(i, j)$, a graph G on $R(i, j) - 1$ vertices that contains neither K_i nor $\overline{K_j}$ as an induced subgraph. Since G is K_i-free, we have that $G \in \mathcal{G}$. This implies that $R_{\mathcal{G}}(i, j) \geq |V(G)| + 1 = R(i, j)$, and hence $R_{\mathcal{G}}(i, j) = R(i, j)$, for all $j \geq 1$. The proof of the second statement in the theorem is identical up to symmetry. □

Note that setting $j = 3$ in Theorem 6 implies the aforementioned result by Matthews on claw-free graphs, and also shows that there are infinitely many nontrivial Ramsey numbers $R_{\mathcal{G}}(i, j)$ that are as hard to determine as $R(i, j)$ when $\mathcal{G}$ is the class of AT-free graphs or the class of P_ℓ-free graphs with $\ell \geq 5$. Setting $i = 3$ shows that the same holds for the class of triangle-free graphs. Recall that for bipartite graphs, an important subclass of triangle-free graphs, we have obtained a formula for determining all Ramsey numbers in Theorem 5. In this section, we show that the same can be achieved for some important subclasses of claw-free graphs.

Recently, Chudnovsky and Seymour [8] proved that every claw-free graph can be composed from graphs belonging to some basic classes, the two principal ones being line graphs and long circular interval graphs [9]. In Section 4.1, we determine all Ramsey numbers for line graphs. In Section 4.2, we do the same for long circular interval graphs, as well as for fuzzy linear interval graphs and fuzzy circular interval graphs, which are all subclasses of claw-free graphs; these classes will be defined in Section 4.2.

4.1 Ramsey numbers for line graphs

For every graph G, the *line graph* of G, denoted $L(G)$, is the graph with vertex set $E(G)$, where there is an edge between two vertices $e, e' \in E(G)$ if and only if the edges e and e' are incident in G. Graph G is called the *preimage graph* of $L(G)$. A graph is a *line graph* if it is the line graph of some graph. Let $\mathcal{L}$ denote the class of all line graphs. In this subsection, we determine all Ramsey numbers for line graphs.

Recall that the value of $R_{\mathcal{L}}(i,j)$ for $i \in \{1,2\}$ and all $j \geq 1$ follow from Observation 1. The case $i = 3$ is the first nontrivial case for the class of line graphs. In his study of the Ramsey numbers $R_{\mathcal{C}}(3,j)$ for all $j \geq 1$, where $\mathcal{C}$ is the class of claw-free graphs, Matthews [18] used arguments that yield the following theorem, which also holds for claw-free graphs. We present its proof for the sake of completeness.

Theorem 7. *For every integer $j \geq 1$, $R_{\mathcal{L}}(3,j) = \lfloor (5j-3)/2 \rfloor$.*

Proof. Suppose G is a line graph that contains neither a clique of size 3 nor an independent set of size j. Since G is both triangle-free and claw-free, we must have $\Delta(G) \leq 2$. Hence G is the disjoint union of a collection of paths and cycles. Let $\mathcal{S}$ be the set of connected components of G, and let $\mathcal{S}_0 \subseteq \mathcal{S}$ be the set of connected components of G that are odd cycles. Then $\alpha(S) = (|V(S)|-1)/2$ for every $S \in \mathcal{S}_0$ and $\alpha(S) = \lceil |V(S)|/2 \rceil$ for every $S \in \mathcal{S} \setminus \mathcal{S}_0$, which implies that

$$\alpha(G) = \sum_{S \in \mathcal{S}_0} (|V(S)|-1)/2 + \sum_{S \in \mathcal{S} \setminus \mathcal{S}_0} \lceil |V(S)|/2 \rceil \geq \sum_{S \in \mathcal{S}} |V(S)|/2 - \sum_{S \in \mathcal{S}_0} 1/2\,,$$

and hence $\alpha(G) \geq (|V(G)| - |\mathcal{S}_0|)/2$.

Since G is triangle-free, every connected component in $\mathcal{S}_0$ contains at least 5 vertices. Hence $|\mathcal{S}_0| \leq |V(G)|/5$, which together with the inequality $\alpha(G) \geq (|V(G)| - |\mathcal{S}_0|)/2$ implies that $\alpha(G) \geq 2|V(G)|/5$. On the other hand, we have that $\alpha(G) \leq j-1$, since we assumed that G has no independent set of size j. Combining these two bounds on $\alpha(G)$ yields the inequality $2|V(G)|/5 \leq j-1$, or equivalently $|V(G)| \leq 5(j-1)/2 \leq \lfloor (5j-3)/2 \rfloor - 1$, where the last inequality holds due to the fact that $|V(G)|$ is an integer. This shows that any line graph that has neither a clique of size 3 nor an independent set of size j has at most $\lfloor (5j-3)/2 \rfloor - 1$ vertices, which implies that $R_{\mathcal{L}}(3,j) \leq \lfloor (5j-3)/2 \rfloor$.

It remains to show that $R_{\mathcal{L}}(3,j) \geq \lfloor (5j-3)/2 \rfloor$ for every $j \geq 1$. In order to do this, it suffices to show that for every $j \geq 1$, there exists a line graph G_j on $\lfloor (5j-3)/2 \rfloor - 1$ vertices satisfying $\omega(G_j) < 3$ and $\alpha(G) < j$. We construct such a graph G_j for every $j \geq 1$ as follows. If $j = 2k$, then G_j is the disjoint union of $k-1$ copies of C_5 and one copy of K_2, while we define G_j to be the disjoint union of k copies of C_5 if $j = 2k+1$. Note that G_j is a line graph for every $j \geq 1$. It is easy to verify that for every $j \geq 1$, it holds that $\omega(G_j) = 2 < 3$ and $\alpha(G_j) = j-1 < j$. If $j = 2k$, then $|V(G_j)| = 5(k-1)+2 = (10k-6)/2 = (5j-4)/2 - 1 = \lfloor (5j-3)/2 \rfloor - 1$, where the last equality holds since j is even. In a similar way, it is easy to check that $|V(G)| = \lfloor (5j-3)/2 \rfloor - 1$ if $j = 2k+1$. We conclude that $R_{\mathcal{L}}(3,j) \geq \lfloor (5j-3)/2 \rfloor$ and consequently $R_{\mathcal{L}}(3,j) = \lfloor (5j-3)/2 \rfloor$ for every $j \geq 1$. □

In order to determine the Ramsey numbers $R_{\mathcal{L}}(i,j)$ for $i \geq 4$, we make use of the following two results, the first of which is an easy observation.

Lemma 2. *Let H be a graph, let $G = L(H)$ be the line graph of H, and let $i \geq 4$ and $j \geq 1$ be two integers. Then H has a vertex of degree at least i if and only if G has a clique of size i. Moreover, H has a matching of size j if and only if G has an independent set of size j.*

Proof. It is clear from the definition of line graphs that if H contains a vertex of degree at least i or a matching of size j, then G contains a clique of size i or an independent set of size j, respectively. For the reverse direction, suppose G has a clique X of size i. Let $e_1, \ldots, e_i$ be the edges in H corresponding to the vertices in X. Since X is a clique and $i \geq 4$, all edges in $\{e_1, \ldots, e_i\}$ must share a common vertex v. Hence H contains a vertex of degree at least i. If G contains a independent set of size j, then the corresponding edges in H form a matching of size j. □

Theorem 8 ([1, 2, 10]). *Let $i \geq 4$ and $j \geq 1$ be two integers, and let H be an arbitrary graph such that $\Delta(H) < i$ and $\nu(H) < j$. Then*

$$|E(H)| \leq \begin{cases} i(j-1)-(t+r)+1 & \text{if } i = 2k \\ i(j-1)-r+1 & \text{if } i = 2k+1, \end{cases}$$

where $j = tk + r$, $t \geq 0$ and $1 \leq r \leq k$, and this bound is tight.

In a preliminary version [2] of our current paper, we presented Theorem 8 with a full proof. Very recently, we have been made aware of the fact that Theorem 8 had already been proved by Balachandran and Khare [1], and that the upper bound in Theorem 8 also appeared in a paper by Chvátal and Hanson (Lemma 2 in [10]).

Lemma 2 and Theorem 8 yield the following formula for all the Ramsey numbers for line graphs that were not covered by Observation 1 and Theorem 7.

Theorem 9. *For every pair of integers $i \geq 4$ and $j \geq 1$, it holds that*

$$R_{\mathcal{L}}(i,j) = \begin{cases} i(j-1)-(t+r)+2 & \text{if } i = 2k \\ i(j-1)-r+2 & \text{if } i = 2k+1, \end{cases}$$

where $j = tk + r$, $t \geq 0$ and $1 \leq r \leq k$.

Proof. For notational convenience, we define a function ρ as follows: for every pair of integer $i \geq 4$ and $j \geq 1$, let

$$\rho(i,j) = \begin{cases} i(j-1)-(t+r)+1 & \text{if } i = 2k \\ i(j-1)-r+1 & \text{if } i = 2k+1, \end{cases}$$

where $j = tk + r$, $t \geq 0$ and $1 \leq r \leq k$. Then an equivalent way of stating Theorem 9 is to say that $R_{\mathcal{L}}(i,j) = \rho(i,j) + 1$ for every $i \geq 4$ and $j \geq 1$.

Let G be a line graph that contains neither a clique of size i nor an independent set of size j, and let H be the preimage graph of G. Then $\Delta(H) < i$ and $\nu(H) < j$ as a result of Lemma 2. Hence, by Theorem 8, we have that $|E(H)| \leq \rho(i,j)$. Since G is the line graph of H, we have $|V(G)| = |E(H)| \leq \rho(i,j)$, implying the upper bound $R_{\mathcal{L}}(i,j) \leq \rho(i,j) + 1$. To prove the matching lower bound, note that Theorem 8 ensures, for every $i \geq 4$ and $j \geq 1$, the existence of a graph H with exactly $\rho(i,j)$ edges that satisfies $\Delta(H) < i$ and $\nu(H) < j$. The line graph $G = L(H)$ of such a graph has exactly $\rho(i,j)$ vertices, and contains neither a clique of size i nor an independent set of size j due to Lemma 2. This implies that $R_{\mathcal{L}}(i,j) \geq \rho(i,j) + 1$, and consequently $R_{\mathcal{L}}(i,j) = \rho(i,j) + 1$, for every $i \geq 4$ an $j \geq 1$. □

4.2 Ramsey numbers for fuzzy circular interval graphs

Given a circle $\mathcal{C}$, a *closed interval* of $\mathcal{C}$ is a proper subset of $\mathcal{C}$ homeomorphic to the closed unit interval $[0,1]$; in particular, every closed interval of $\mathcal{C}$ has two distinct endpoints. Linear interval graphs and circular interval graphs, defined by Chudnovsky and Seymour [8] as two basic classes of claw-free graphs, can be defined as follows.

Definition 1. *A graph $G = (V, E)$ is a* circular interval graph *if the following conditions hold:*

- *there is an injective mapping φ from V to a circle $\mathcal{C}$;*
- *there is a set $\mathcal{I}$ of closed intervals of $\mathcal{C}$, none including another, such that two vertices $u, v \in V$ are adjacent if and only if $\varphi(u)$ and $\varphi(v)$ belong to a common interval of $\mathcal{I}$.*

A graph G is a linear interval graph *if it satisfies the above conditions when we substitute "circle" by "line".*

We call the triple $(V, \varphi, \mathcal{I})$ in Definition 1 a *circular interval model* of G. A graph $G = (V, E)$ is a *long circular interval graph* if it has a circular interval model $(V, \varphi, \mathcal{I})$ such that no three intervals in $\mathcal{I}$ cover the entire circle $\mathcal{C}$.

It is known that linear interval graphs and circular interval graphs are equivalent to proper interval graphs and proper circular-arc graphs, respectively [8]. It is immediate from the above definitions that circular interval graphs form a superclass of both long circular interval graphs and linear interval graphs. We now define a superclass of circular interval graphs that was introduced by Chudnovsky and Seymour [8] as yet another important class of claw-free graphs.

Definition 2. *A graph $G = (V, E)$ is a* fuzzy circular interval graph *if the following conditions hold:*

- *there is a (not necessarily injective) mapping φ from V to a circle $\mathcal{C}$;*
- *there is a set $\mathcal{I}$ of closed intervals of $\mathcal{C}$, none including another, such that no point of $\mathcal{C}$ is an endpoint of more than one interval in $\mathcal{I}$, and*
  - *if two vertices $u, v \in V$ are adjacent, then $\varphi(u)$ and $\varphi(v)$ belong to a common interval of $\mathcal{I}$;*
  - *if two vertices $u, v \in V$ are not adjacent, then either there is no interval in $\mathcal{I}$ that contains both $\varphi(u)$ and $\varphi(v)$, or there is exactly one interval in $\mathcal{I}$ whose endpoints are $\varphi(u)$ and $\varphi(v)$.*

A graph G is a fuzzy linear interval graph *if it satisfies the above conditions when we substitute "circle" by "line".*

We call the triple $(V, \varphi, \mathcal{I})$ in Definition 2 a *fuzzy circular interval model* (or simply *model*) of G, and call $(V, \varphi, \mathcal{I})$ a *fuzzy linear interval model* if $\mathcal{I}$ is a set of intervals of a line. Clearly, the class of fuzzy linear interval graphs is a subclass of fuzzy circular interval graphs. This also holds for the class of circular interval graphs (and hence for the class of proper circular-arc graphs), as they are exactly those fuzzy circular interval graphs $G = (V, E)$ that have a model $(V, \varphi, \mathcal{I})$ such that φ is injective [8]. Similarly, a graph $G = (V, E)$ is a linear interval graph if and only if it is a fuzzy linear interval graph that has a model $(V, \varphi, \mathcal{C})$ such that φ is injective.

Let us remark that *circular-arc graphs* form neither a subclass nor a superclass of fuzzy circular interval graphs: the claw is an example of a circular-arc graph that is not a fuzzy circular interval graph, whereas the complement of C_6 is known not to be a circular-arc graph (see, e.g., [3]), but is a fuzzy circular interval graph (see Figure 2 for a fuzzy circular interval model of this graph).

Recall that the class of linear interval graphs coincides with the class of proper interval graphs, and that we determined all Ramsey numbers for proper interval graphs in Theorem 2. We now show that the formula in Theorem 2 can also be used to determine all Ramsey numbers for fuzzy linear interval graphs. Very recently, Chudnovsky and Plumettaz [6] proved that every linear interval trigraph is perfect, where linear interval trigraph is a notion closely related to linear interval graphs. Their argument can be adapted to prove the following lemma.

Lemma 3. *Every fuzzy linear interval graph is perfect.*

Proof. We prove the lemma by induction on the number of vertices. Note that the graph on one vertex is perfect. Suppose that every fuzzy linear interval graph on at most $n-1$ vertices is perfect, and let $G = (V, E)$ be a fuzzy linear interval graph on n vertices with model $(V, \varphi, \mathcal{I})$,

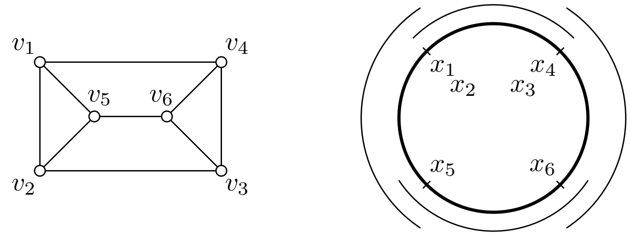

Fig. 2. The graph $\overline{C_6}$ and a fuzzy circular interval model of this graph, where $x_i = \varphi(v_i)$ for every $i \in \{1, \ldots, 6\}$.

where $\mathcal{I}$ is a set of intervals of a line $\mathcal{L}$. For any two vertices $u, v \in V$, we write $\varphi(u) \leq \varphi(v)$ if $\varphi(u) = \varphi(v)$ or if $\varphi(u)$ lies to the left of $\varphi(v)$ on the line $\mathcal{L}$. Let $v_n \in V$ be such that $\varphi(v) \leq \varphi(v_n)$ for all $v \in V \setminus \{v_n\}$, i.e., no vertex of G is mapped to the right of v_n. Let $w \in N_G(v_n)$ be such that $\varphi(w) \leq \varphi(v)$ for all $v \in N_G(v_n)$, i.e., no neighbor of v_n is mapped to the left of w. Since w is adjacent to v_n, there is an interval in $I \in \mathcal{I}$ that contains both w and v_n. By the definition of w, every vertex of $N_G(v_n)$ belongs to I, which implies that $N_G(v_n)$ is a clique in G.

Due to the strong perfect graph theorem [7], in order to complete the proof of the lemma, it suffices to prove that G contains neither an odd hole nor an odd antihole. For contradiction, suppose there exists a set $X \subseteq V$ such that X induces an odd hole or an odd antihole. Note that for every $x \in X$, the set $N_G(x) \cap X$ is not a clique in G. Since we proved that $N_G(v_n)$ is a clique, we deduce that $v_n \notin X$. Consequently, X is a subset of the vertices of the graph $G - v_n$, which means that $G - v_n$ contains an odd hole or an odd antihole. However, by the induction hypothesis and the fact that $G - v_n$ is a fuzzy linear interval graph, the graph $G - v_n$ is perfect. This yields the desired contradiction. $\square$

It is easy to see that any disjoint union of complete graphs is a fuzzy linear interval graph. Hence Theorem 2 readily implies the next result.

Theorem 10. *Let $\mathcal{G}$ be the class of fuzzy linear interval graphs. Then $R_{\mathcal{G}}(i,j) = (i-1)(j-1)+1$ for all $i, j \geq 1$.*

Before we proceed, we need some additional terminology and settle notation. Let $G = (V, E)$ be a fuzzy circular interval graph with model $(V, \varphi, \mathcal{I})$. For every point p on the circle $\mathcal{C}$, we define $\varphi^{-1}(p) = \{v \in V \mid \varphi(v) = p\}$. Moreover, for every interval I of $\mathcal{C}$ (possibly $I \notin \mathcal{I}$), we define $\varphi^{-1}(I) = \bigcup_{p \in I} \varphi^{-1}(p)$. Throughout this section, we assume that for every $I \in \mathcal{I}$, the set $\varphi^{-1}(I)$ is non-empty, as otherwise we can simply delete I from $\mathcal{I}$. For any two points p_1 and p_2 on the circle $\mathcal{C}$, we write $[p_1, p_2]$ to denote the closed interval of $\mathcal{C}$ that we span when we traverse $\mathcal{C}$ clockwise from p_1 to p_2. We write $\langle p_1, p_2 \rangle = [p_1, p_2] \setminus \{p_1, p_2\}$ and $\langle p_1, p_2] = [p_1, p_2] \setminus \{p_1\}$ and $[p_1, p_2\rangle = [p_1, p_2] \setminus \{p_2\}$.

We would like to point out that every fuzzy circular interval graph $G = (V, E)$ has a model $(V, \varphi, \mathcal{I})$ such that for every point p on $\mathcal{C}$ for which the set $X_p = \varphi^{-1}(p)$ is non-empty, the vertices of X_p form a clique in G. To see this, note that if there is an edge between any two vertices in X_p, then there is an interval of $\mathcal{I}$ that covers p, and hence X_p is a clique. If the vertices in X_p form an independent set in G, then there is an interval $[p^-, p^+]$ of $\mathcal{C}$ that contains p and does not intersect with any interval in $\mathcal{I}$. Hence we can simply define a new model $(V, \varphi', \mathcal{I})$ such that φ' maps the vertices of X_p to distinct points in the interval $[p^-, p^+]$.

Let I be a closed interval of a circle $\mathcal{C}$. We write I^ℓ and I^r to denote the two points on $\mathcal{C}$ such that $I = [I^\ell, I^r]$. By slight abuse of terminology, we refer to I^ℓ and I^r as the *left endpoint* and the *right endpoint* of the interval I, respectively. A point p is an *interior point* of I if p is not an endpoint of I, i.e., if $p \in I \setminus \{I^\ell, I^r\}$. For any two points p_1 and p_2 that belong to I, we write $p_1 < p_2$ if p_2 does not belong to the subinterval $[I^\ell, p_1]$, i.e., if p_1 is closer to the left endpoint of I than p_2 is.

In order to determine all Ramsey numbers for long circular interval graphs and fuzzy circular interval graphs, we will use the following two lemmas.

Lemma 4. *Every fuzzy circular interval graph G contains a clique X of size at most $\omega(G) - 1$ such that $G - X$ is a fuzzy linear interval graph.*

Proof. Let $G = (V, E)$ be a fuzzy circular interval graph with model $(V, \varphi, \mathcal{I})$, where $\mathcal{I}$ is a set of intervals of a circle $\mathcal{C}$. The lemma trivially holds if G is a fuzzy linear interval graph. Suppose G is not a fuzzy linear interval graph. Let P be the set of $2|\mathcal{I}|$ points on the circle $\mathcal{C}$ that are endpoints of intervals in $\mathcal{I}$. We partition P into two sets by defining $P_\ell = \{p \in P \mid p = I^\ell \text{ for some } I \in \mathcal{I}\}$ and $P_r = \{p \in P \mid p = I^r \text{ for some } I \in \mathcal{I}\}$. We also define $Q = \{q \in \mathcal{C} \mid \varphi^{-1}(q) \neq \emptyset\}$, i.e., Q consists of the points q on $\mathcal{C}$ such that φ maps at least one vertex of V to q.

Now let $a \in Q$ be an arbitrary point on the circle. We claim that a is an interior point of at least one interval of $\mathcal{I}$. Since G is not a fuzzy linear interval graph, a is covered by an interval $J \in \mathcal{I}$. Suppose a is not an interior point of J, and without loss of generality assume that $a = J^\ell$. Let $p \in P \setminus \{a\}$ be the first point of P that we encounter when we traverse $\mathcal{C}$ counterclockwise from a, i.e., p is the unique point in $P \setminus \{a\}$ such that the interval $\langle p, a\rangle$ contains no vertex of P. Let $q \in \langle p, a\rangle$. Since G is not a fuzzy linear interval graph, q is covered by an interval $J' \in \mathcal{I}$. Since $q \notin P$ and $a = J^\ell$, and no two intervals in $\mathcal{I}$ have a common endpoint by definition, it holds that $J' \neq J$ and hence a is an interior point of J'.

Let $\mathcal{I}_a \subseteq \mathcal{I}$ consist of all the intervals in $\mathcal{I}$ that contain a as an interior point, where $\mathcal{I}_a = \{A_1, A_2, \ldots, A_s\}$ such that $a < A_1^r < A_2^r < \cdots < A_s^r$. As we argued above, the set $\mathcal{I}_a$ is non-empty. If a happens to be the endpoint of some interval $J \in \mathcal{I} \setminus \mathcal{I}_a$, then we assume, without loss of generality, that $a = J^\ell$. We define $X = \varphi^{-1}([A_1^\ell, a\rangle)$, i.e., X consists of those vertices of G that are mapped by φ to some point on $\mathcal{C}$ in the interval $[A_1^\ell, a\rangle$. Since the interval $[A_1^\ell, a\rangle$ is contained in the interval A_1, this set X is a clique. Moreover, since $[A_1^\ell, a]$ is also a subinterval of A_1, the set $X \cup \varphi^{-1}(a)$ is also a clique in G. Since $a \in Q$, the set $\varphi^{-1}(a)$ is non-empty, so X has size at most $\omega(G) - 1$.

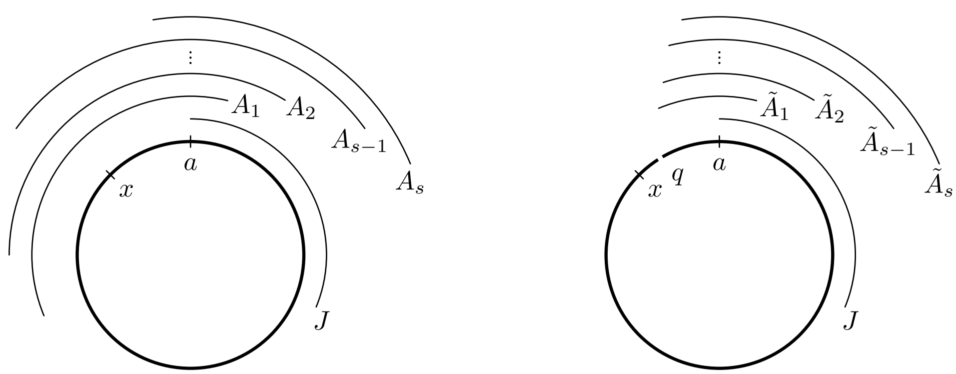

Fig. 3. The intervals $\tilde{A}_1, \ldots, \tilde{A}_s$ are obtained from the intervals $A_1, \ldots, A_s$ by moving the left endpoints of $A_1, \ldots, A_s$ to within the interval $\langle x, a\rangle$. The resulting fuzzy circular interval model can be modified into a fuzzy linear interval model by cutting the circle at any point q in the interval $\langle x, \tilde{A}_1^\ell\rangle$.

It remains to prove that the graph $G - X$ is a fuzzy linear interval graph. We do this by constructing a fuzzy linear interval model $(V \setminus X, \varphi', \mathcal{I}')$ of $G - X$ from the model $(V, \varphi, \mathcal{I})$ of G as follows (see Figure 3 for a helpful illustration). First, we define φ' to be the restriction of φ to the vertices of $V \setminus X$, i.e., φ' is the mapping from $V \setminus X$ to $\mathcal{C}$ such that $\varphi'(v) = \varphi(v)$ for all $v \in V \setminus X$. Clearly, $(V \setminus X, \varphi', \mathcal{I})$ is a fuzzy circular interval model of $G - X$. Let $x \in P_r \cup Q$ be such that the interval $\langle x, a \rangle$ does not contain any element of $P_r \cup Q$. Note that it is possible that the interval $\langle x, a \rangle$ contains an element of P_ℓ; any such element is the left endpoint of some interval in $\mathcal{I}_a$ (for example, the left endpoint of interval A_s in Figure 3 lies in the interval $\langle x, a \rangle$). Informally speaking, we will now "shrink" the intervals in $\mathcal{I}_a$ by moving their left endpoints in such a way that all these left endpoints end up in the interval $\langle x, a \rangle$ and the obtained model is still a model of $G - X$, i.e., the new intervals force the same adjacencies and non-adjacencies in the corresponding graph.

Formally, we define, for every $p \in \{1, \ldots, s\}$, a new closed interval $\tilde{A}_p$ of $\mathcal{C}$ such that $\tilde{A}_p^r = A_p^r$ and $\tilde{A}_p^\ell$ is chosen arbitrarily in such a way that $x < \tilde{A}_1^\ell < \tilde{A}_2^\ell < \cdots < \tilde{A}_s^\ell < a$; see Figure 3. Let $\tilde{\mathcal{I}}_a = \{\tilde{A}_1, \ldots, \tilde{A}_s\}$, and let $\mathcal{I}' = (\mathcal{I} \setminus \mathcal{I}_a) \cup \tilde{\mathcal{I}}_a$. Let us first show that $(V \setminus X, \varphi', \mathcal{I}')$ is a fuzzy circular interval model of $G - X$. First note that we chose the left endpoints of the intervals in $\tilde{\mathcal{I}}_a$ in such a way that no interval of $\tilde{\mathcal{I}}_a$ contains another. Moreover, since the interval $\langle x, a \rangle$ contains no vertex of P_r, no point of $\mathcal{C}$ is an endpoint of more than one interval in $\mathcal{I}'$. From the definition of X it follows that, for every $p \in \{1, \ldots, s\}$, φ does not map any vertex of $G - X$ to a point in the interval $[A_p^\ell, a\rangle$. In other words, for every vertex v of $G - X$, interval A_p contains the point $\varphi'(v)$ if and only if interval $\tilde{A}_p$ does, for every $p \in \{1, \ldots, s\}$. This guarantees that the triple $(V \setminus X, \varphi', \mathcal{I}')$ indeed is a fuzzy circular interval model of $G - X$. To see why $(V \setminus X, \varphi', \mathcal{I}')$ is a fuzzy *linear* interval model of $G - X$, it suffices to observe that we can cut the circle $\mathcal{C}$ at any point q in the interval $\langle x, \tilde{A}_1^\ell \rangle$, as any such point q is not covered by any interval in $\mathcal{I}'$ (again, see Figure 3). This completes the proof of Lemma 4. □

As the next observation will be used also in the next section, we state it as a separate lemma.

Lemma 5. *Let G be a graph such that $\omega(G) < i$ and $\alpha(G) < j$ for two integers $i, j \geq 3$. If G contains a clique X of size at most $\omega(G) - 1$ such that $G - X$ is a perfect graph, then G has at most $(i-1)j - 1$ vertices.*

Proof. Suppose G contains a clique X such that $|X| \leq \omega(G) - 1 \leq i - 2$ and $G - X$ is perfect. Since $G - X$ is an induced subgraph of G, it contains neither a clique of size i nor an independent set of size j. Hence, due to Theorem 2, we have that $|V(G - X)| \leq (i-1)(j-1)$. Then $|V| = |V(G - X)| + |X| \leq (i-1)(j-1) + (i-2) = (i-1)j - 1$. □

We are now ready to determine all Ramsey numbers for long circular interval graphs and fuzzy circular interval graphs. Since the class of long circular interval graphs (and hence its superclass of fuzzy circular interval graphs) contains all edgeless graphs and all complete graphs, Observations 1 and 2 yield the Ramsey numbers for both classes for all $i, j \in \{1, 2\}$. All other Ramsey numbers for these two classes are given by the following formula.

Theorem 11. *Let $\mathcal{G}$ be the class of long circular interval graphs or the class of fuzzy circular interval graphs. Then $R_{\mathcal{G}}(i, j) = (i-1)j$ for all $i, j \geq 3$.*

Proof. Let G be a fuzzy circular interval graph, and let i and j be two integers such that $i, j \geq 3$. Suppose G contains neither a clique of size i nor an independent set of size j. By Lemma 4, G contains a clique X of size at most $i - 2$ such that $G - X$ is a fuzzy linear interval graph. Since the graph $G - X$ is perfect due to Lemma 3, we know that G has at most $(i-1)j - 1$ vertices as a result of Lemma 5. Hence $R_{\mathcal{G}}(i, j) \leq (i-1)j$ for all $i, j \geq 3$ if $\mathcal{G}$ is the class of fuzzy circular

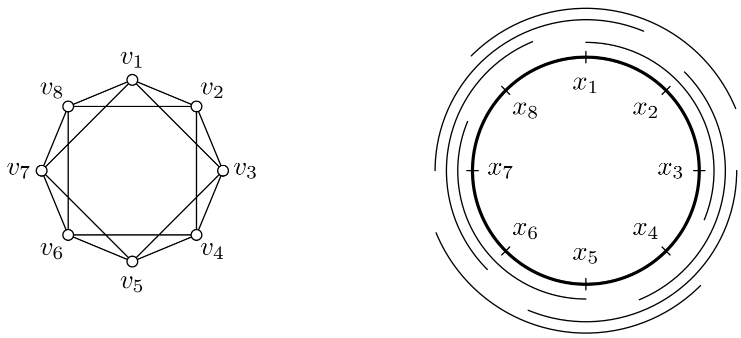

Fig. 4. The graph $G^*_{i,j}$ when $i = 4$ and $j = 3$, together with a circular interval model in which no three intervals cover the circle.

interval graphs, and the same holds if $\mathcal{G}$ is the class of long circular interval graphs, as they form a subclass of fuzzy circular interval graphs.

It remains to prove that $R_{\mathcal{G}}(i, j) \geq (i-1)j$ for all $i, j \geq 3$. Note that it suffices to construct a long circular interval graph on $n = (i-1)j - 1$ vertices that has no clique of size i and no independent set of size j. For every $i, j \geq 3$, let $G^*_{i,j}$ be the $(i-2)$th power of $C_{(i-1)j-1}$, i.e., let $G^*_{i,j} = (V, E)$ be the graph obtained from a cycle C on $n = (i-1)j - 1$ vertices by making any $i-1$ consecutive vertices of cycle into a clique. For any subset S of vertices in $G^*_{i,j}$, we say that the vertices of S are consecutive if they appear consecutively on the cycle C. To show that $G^*_{i,j}$ is a long circular interval graph, we construct a long circular interval model $(V, \varphi, \mathcal{I})$ of $G^*_{i,j}$ as follows (see Figure 4 for an illustration of the case where $i = 4$ and $j = 3$). Let $V = \{v_1, v_2, \ldots, v_n\}$ and let $\varphi : V \to \mathcal{C}$ be a mapping that injectively maps the vertices of V to the circle $\mathcal{C}$ in such a way that $\varphi(v_1), \ldots, \varphi(v_n)$ appear consecutively on the circle in clockwise order. Let $x_i = \varphi(v_i)$ for every $i \in \{1, \ldots, n\}$, and let $X = \{x_1, \ldots, x_n\}$. For every $p \in \{1, \ldots, n\}$, we define an interval I_p such that $I_p^\ell = x_p$ and I_p^r is chosen arbitrarily such that $x_{p+i-1} < I_p^r < x_{p+i}$, where the indices are taken modulo n (See Figure 4). Let $\mathcal{I} = \{I_1, \ldots, I_n\}$.

Since every interval in $\mathcal{I}$ contains the image of exactly $i-1$ consecutive vertices of $G^*_{i,j}$, forcing them to be in a clique, the triple $(V, \varphi, \mathcal{I})$ clearly is a circular interval model of $G^*_{i,j}$. To prove that $G^*_{i,j}$ is a *long* circular interval graph, it suffices to argue that no three intervals of $\mathcal{I}$ cover the entire circle. By construction, any two intervals $I_p, I_q \in \mathcal{I}$ overlap if and only if there exists a point $x \in X$ that is contained in both I_p and I_q. As a result, any three intervals of $\mathcal{I}$ cover at most $3(i-1) - 2$ points of X. Recall that $j \geq 3$, so $|X| \geq 3(i-1) - 1$. This implies that for any three intervals in $\mathcal{I}$, at least one point of X is not covered by these three intervals.

It is clear that $G^*_{i,j}$ contains no clique of size i. To show that $\alpha(G^*_{i,j}) < j$, suppose, for contradiction, that $G^*_{i,j}$ contains an independent set S of size j. Since every $i-1$ consecutive vertices in $G^*_{i,j}$ form a clique, we have at least $i-2$ consecutive vertices of $V(G^*_{i,j}) \setminus S$ between any two vertices of S. Since $|S| \geq j$, this implies that $G^*_{i,j}$ contains at least $(i-2)j + |S| \geq (i-1)j$ vertices. This contradiction to the fact that $|V(G^*_{i,j})| = (i-1)j - 1$ completes the proof. $\square$

5 Ramsey numbers for circular-arc graphs and cactus graphs

5.1 Ramsey numbers for circular-arc graphs

In this subsection, we determine all Ramsey numbers for circular-arc graphs and proper circular-arc graphs. We will use the same approach as for fuzzy circular interval graphs. In particular, we will use the following lemma, which strongly resembles Lemma 4, in combination with Lemma 5.

Lemma 6. *Every circular-arc graph G contains a clique X of size at most $\omega(G)-1$ such that $G-X$ is an interval graph.*

Proof. Let $G=(V,E)$ be a circular-arc graph with circular-arc model $(G,\mathcal{I})$, where $\mathcal{I}$ is a set of closed intervals (arcs) of a circle $\mathcal{C}$. Without loss of generality, we assume that no two intervals in $\mathcal{I}$ share an endpoint. Recall that the vertices of G correspond to the intervals in $\mathcal{I}$, and not to points of the circle $\mathcal{C}$ as is the case in the definition of (fuzzy) circular interval graphs. We first show that there is a point q on $\mathcal{C}$ such that at most $\omega(G)-1$ intervals of $\mathcal{I}$ contain q. Let $I \in \mathcal{I}$, and let a be an endpoint of I. If at most $\omega(G)-1$ intervals of $\mathcal{I}$ contain a, then we can simply take $q=a$. Suppose exactly $\omega(G)$ intervals of $\mathcal{I}$ contain a. Let P be the set of $2|V|$ points on $\mathcal{C}$ that are endpoints of intervals in $\mathcal{I}$. Let $p \in P \setminus \{a\}$ be such that p does not belong to I, and there is no point of $P \setminus \{a,p\}$ that lies between a and p. Then we can choose q to be any point on the circle between a and p; any such point q is contained in exactly $\omega(G)-1$ intervals of $\mathcal{I}$, namely all those that contain a, apart from interval I. Let $\mathcal{I}_q \subseteq \mathcal{I}$ denote the subset of intervals containing q.

Now let X denote the subset of vertices of G corresponding to the intervals containing q. It is clear that X is a clique of size at most $\omega(G)-1$. Moreover, the graph $G-X$ is an interval graph, as we can obtain an interval model of $G-X$ from the circular-arc model $(G,\mathcal{I}\setminus\mathcal{I}_q)$ by cutting the circle $\mathcal{C}$ at point q to obtain a line. □

We now determine all Ramsey numbers for circular-arc graphs and proper circular-arc graphs that were not covered by Observations 1 and 2.

Theorem 12. *Let $\mathcal{G}$ be the class of circular-arc graphs or the class of proper circular-arc graphs. Then $R_{\mathcal{G}}=(i-1)j$ for all $i,j \geq 3$.*

Proof. Let $G=(V,E)$ be a circular-arc graph, and let i and j be two integers such that $i,j \geq 3$. Suppose $\omega(G)<i$ and $\alpha(G)<j$. Let $X \subseteq V$ be a clique of size at most $i-2$ such that $G-X$ is an interval graph; the existence of such a clique is guaranteed by Lemma 6. Since all interval graphs are perfect, Lemma 5 implies that G has at most $(i-1)j-1$ vertices. Consequently, we have that $R_{\mathcal{G}}(i,j) \leq (i-1)j$ if $\mathcal{G}$ is the class of circular-arc graphs. Since proper circular-arc graphs form a subclass of circular-arc graphs, the same trivially holds if $\mathcal{G}$ is the class of proper circular-arc graphs. The matching lower bound follows immediately from the fact that the graph $G^*_{i,j}$, constructed in the proof of Theorem 11, is a circular interval graph, and hence a proper circular-arc graph, on $(i-1)j-1$ vertices with $\omega(G^*_{i,j})<i$ and $\alpha(G^*_{i,j})<j$. □

5.2 Ramsey numbers for cactus graphs

In this subsection, we determine all Ramsey numbers for cactus graphs. There exist several equivalent definitions of cactus graphs in the literature. Before we give the definition that will be used in the proof of Theorem 13 below, we first recall some terminology.

Let $G=(V,E)$ be a graph. A *cut vertex* of G is a vertex whose deletion strictly increases the number of connected components. A maximal connected subgraph without a cut vertex is called a *block*. Let $G = (V, E)$ be a connected graph, let $A \subseteq V$ be the set of cut vertices in G, and let $\mathcal{B}$ be the set of blocks of G. The *block graph* of G is the bipartite graph with vertex set $A \cup \mathcal{B}$ such that there is an edge between two vertices $a \in A$ and $B \in \mathcal{B}$ if and only if $a \in V(B)$. It is well-known that the block graph of a connected graph is a tree [11]. We define the *block forest* of a graph to be the disjoint union of the block graphs of its connected components.

A graph G is a *cactus graph* if every block of G with more than two vertices is a cycle.[1] Equivalently, a graph G is a cactus graph if every edge of G is contained in at most one cycle. The class of cactus graphs forms a subclass of planar graphs that contains all forests.

Theorem 13. *Let $\mathcal{G}$ be the class of cactus graphs. Then*

$$R_{\mathcal{G}}(i,j) = \begin{cases} \lfloor \frac{5}{2}(j-1) \rfloor + 1 & \text{if } i = 3 \\ 3(j-1)+1 & \text{if } i \geq 4 \end{cases}$$

for every pair of integers $i \geq 3$ and $j \geq 1$.

Proof. It is well-known that every cactus graph is outerplanar, and that every outerplanar graph G is 3-colorable [19]. Hence the function f, defined by $f(x) = 3$ for every $x \in \mathbb{N}$, is a χ-bounding function for the class of cactus graphs, and Lemma 1 implies that $R_{\mathcal{G}}(i,j) \leq 3(j-1)+1$ for all $i, j \geq 1$. For every $i \geq 4$ and $j \geq 1$, the disjoint union of $j-1$ triangles is a cactus graph on $3(j-1)$ vertices that has neither a clique of size i nor an independent set of size j. Consequently, $R_{\mathcal{G}}(i,j) \geq 3(j-1)+1$, and hence $R_{\mathcal{G}}(i,j) = 3(j-1)+1$, for every $i \geq 4$ and $j \geq 1$.

Now suppose $i = 3$. We show, by induction on j, that for every $j \geq 1$, every cactus graph that contains neither a clique of size 3 nor an independent set of size j has at most $\lfloor \frac{5}{2}(j-1) \rfloor$ vertices. The statement trivially holds when $j = 1$. Suppose G is a cactus graph that contains neither a clique of size 3 nor an independent set of size j for some $j \geq 2$. If G has no edges, then $|V(G)| \leq j-1 \leq \lfloor \frac{5}{2}(j-1) \rfloor$, so we assume that G has at least one edge. Then G has a block B on at least two vertices such that B has degree at most 1 in the block forest of G. This block B is either a connected component of G, or B contains exactly one cut vertex b that has neighbors in the graph $G' = G - V(B)$. In either case, since G is triangle-free, B has an independent set S of size $\lfloor |V(B)|/2 \rfloor$ such that S does not contain b. Then G' does not contain an independent set of size $j - \lfloor |V(B)|/2 \rfloor$, as otherwise the union of this set and S would be an independent set of size j in G. Since $\lfloor |V(B)/2| \rfloor \geq 1$, the induction hypothesis guarantees that G' has at most $\lfloor \frac{5}{2}(j-1-\lfloor |V(B)|/2 \rfloor) \rfloor$ vertices. Consequently, $|V(G)| = |V(G')| + |V(B)| \leq \lfloor \frac{5}{2}(j-1-\lfloor |V(B)|/2 \rfloor) \rfloor + |V(B)| = \lfloor \frac{5}{2}(j-1) - \frac{5}{2}\lfloor |V(B)|/2 \rfloor + |V(B)| \rfloor$. Recall that G is triangle-free, so $|V(B)| \neq 3$. Since $\frac{5}{2}\lfloor |V(B)|/2 \rfloor + |V(B)| \leq 0$ for every $|V(B)| \geq 2$ with $|V(B)| \neq 3$, we find that $|V(G)| \leq \lfloor \frac{5}{2}(j-1) \rfloor$. This implies that $R_{\mathcal{G}}(3,j) \leq \lfloor \frac{5}{2}(j-1) \rfloor + 1$ for every $j \geq 1$.

It remains to prove the matching lower bound. In the proof of Theorem 7, we constructed, for every integer $j \geq 1$, a line graph G_j on $\lfloor (5j-3)/2 \rfloor - 1 = \lfloor \frac{5}{2}(j-1) \rfloor$ vertices satisfying $\omega(G_j) < 3$ and $\alpha(G_j) < j$. Since G_j is a disjoint union of copies of C_5 and possibly one copy of K_2, it is clear that G_j is a cactus graph for every $j \geq 1$. This implies that $R_{\mathcal{G}}(3,j) \geq \lfloor \frac{5}{2}(j-1) \rfloor + 1$ for every $j \geq 1$. □

6 Conclusion

Given the difficulty of determining all Ramsey numbers for claw-free graphs [18] and our positive results on several important subclasses of claw-free graphs, another interesting class to consider

[1] In some definitions, a cactus graph is required to be connected. We do not impose this restriction on cactus graphs, but point out that Theorem 13 would still hold if we did; to see this, it suffices to observe that the graph G_j, constructed in the proof of Theorem 13, can easily be turned into a *connected* cactus graph by making an arbitrarily chosen vertex in one connected component adjacent to exactly one vertex in each of the other connected components.

is the class of quasi-line graphs. A graph is a *quasi-line graph* if the neighborhood of every vertex can be covered by two cliques. Quasi-line graphs form a subclass of claw-free graphs [8] and a superclass of both line graphs and fuzzy circular interval graphs. Is it possible to determine all Ramsey numbers for quasi-line graphs?

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